RANDOM PLANAR GEOMETRY, LENT 2020, EXAMPLE SHEET 2

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Problem 1. Suppose that $\mathbf{e}$ is a Brownian excursion. Fix $t_0 \in [0,1)$. For each $r \geq 0$, we let $\overline{r}$ be the fractional part of r given by $r - \lfloor r \rfloor$. Define the process $\widetilde{\mathbf{e}} \colon [0,1] \to \mathbf{R}_+$ by setting

$$\widetilde{\mathbf{e}}(t) = \mathbf{e}(t_0) + \mathbf{e}(\overline{t_0 + t}) - 2m_{\mathbf{e}}(t_0, \overline{t_0 + t})$$

where $m_{\mathbf{e}}(s,t) = \inf_{r \in [s \wedge t, s \vee t]} \mathbf{e}(r)$. Show that $\widetilde{\mathbf{e}}$ is a Brownian excursion. *[Hint: show that the law of a uniformly random element of* $\mathbf{LT}_k$ *is invariant under translating its root and deduce that a simple random walk excursion satisfies an analogous property. Conclude by taking a scaling limit.]*

Problem 2. Suppose that $(\mathbf{e}, Z)$ is a Brownian snake. Suppose that we have the setup and define $\widetilde{\mathbf{e}}$ as in the previous problem. Let also

$$\widetilde{Z}_t = Z_{\overline{t+t_0}} - Z_{t_0}.$$

Show that $(\widetilde{\mathbf{e}}, \widetilde{Z})$ is a Brownian snake.

Problem 3. Suppose that $(\ell, \tau, \epsilon) \in \mathbf{LT}_n \times \{-1, 1\}$. Let $q \in \mathcal{Q}_n^\bullet$ be given by $q = Q(\ell, \tau, \epsilon)$. Suppose that $u, v \in \mathbf{V}(q) \setminus \{v_*\}$ and let e, e' be corners of τ such that $e^- = u$ and $(e^-)' = v$. Show that

$$d(u,v) \leq \ell(u) + \ell(v) - 2 \min_{e'' \in [e, e']} \ell(e'') + 2$$

where $[e, e']$ are the corners in the contour exploration from e to e' and d is the graph distance on q.

Problem 4. Assume that we have the same setup as in Problem 3. Show that

$$d(u,v) \geq \ell(u) + \ell(v) - 2 \min_{w \in [[u,v]]} \ell(w)$$

where $[[u,v]]$ is the set of all vertices lying on the geodesic path from u to v in τ.

Problem 5. Suppose that $g \colon [0,1] \to \mathbf{R}_+$ is continuous with $g(0) = g(1) = 0$. For each $s, t \in [0,1]$, let $m_g(s,t) = \inf_{r \in [s \wedge t, s \vee t]} g(r)$. Show that for every $s_1, \ldots, s_n \in [0,1]$ and $\lambda_1, \ldots, \lambda_n \in \mathbf{R}$ we have that

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j m_g(s_i, s_j) \geq 0.$$

Problem 6. Suppose that (ℓ_k, τ_k) is uniformly distributed on $\mathbf{LT}_k$ and let V_k be the label contour function. Show that for each $k \in \mathbf{N}$ and $s, t \in [0,1]$ we have that

$$\mathbf{E}\left[\left(\frac{V_k(2kt) - V_k(2ks)}{k^{1/4}} \right)^{4p} \right] \leq c_p |t - s|^p.$$

Conclude that the sequence of functions $t \mapsto V_k(2kt)/k^{1/4}$ is tight.

Problem 7. Convince yourself that $Q_{\mathrm{CVS}}(T_{\mathrm{CVS}}(q)) = q$ for all $q \in \mathcal{Q}_n^\bullet$.

Problem 8. Establish the following properties of the half-plane capacity (hcap).

(i) If $r > 0$, $A \in \mathcal{Q}$, then $\mathrm{hcap}(rA) = r^2 \mathrm{hcap}(A)$ and $g_{rA}(z) = r g_A(z/r)$.
(ii) If $x \in \mathbf{R}$, $A \in \mathcal{Q}$, then $\mathrm{hcap}(A+x) = \mathrm{hcap}(A)$ and $g_{A+x}(z) = g_A(z-x)+x$.
(iii) If $A, \widetilde{A} \in \mathcal{Q}$ with $A \subseteq \widetilde{A}$ then

$$\mathrm{hcap}(\widetilde{A}) = \mathrm{hcap}(A) + \mathrm{hcap}(g_A(\widetilde{A} \setminus A)).$$

Problem 9.

(i) Show that $f(z) = z + 1/z$ is a conformal transformation from $\mathbf{H} \setminus \overline{\mathbf{D}}$ to $\mathbf{H}$.
(ii) Using the conformal invariance of Brownian motion, show that the hitting density (with respect to Lebesgue measure) for a complex Brownian motion starting from $z \in \mathbf{H}$ on the real line $\partial \mathbf{H}$ is given by

$$p(z,u) = \frac{1}{\pi} \frac{y}{(x-u)^2 + y^2} \quad \text{where} \quad z = x+iy, \quad u \in \partial \mathbf{H}.$$

(Note that $p(i,\cdot)$ is the Cauchy distribution on $\mathbf{R}$.)
(iii) Using the conformal invariance of Brownian motion, show that the density $p(z, e^{i\theta})$, $\theta \in [0,\pi]$, for the first exit distribution (with respect to Lebesgue measure) of a complex Brownian motion on $\mathbf{H} \cap \partial \mathbf{D}$ starting from $z \in \mathbf{H} \setminus \overline{\mathbf{D}}$ satisfies:

$$p(z, e^{i\theta}) = \frac{2}{\pi} \frac{\mathrm{Im}(z)}{|z|^2} \sin(\theta) \left(1 + O(|z|^{-1})\right) \quad \text{as} \quad z \to \infty.$$

Problem 10. Using the previous problem, show that if $A \in \mathcal{Q}$ with $A \subseteq \overline{\mathbf{D}} \cap \mathbf{H}$ then

$$\mathrm{hcap}(A) = \frac{2}{\pi} \int_0^\pi \mathbf{E}_{e^{i\theta}}[\mathrm{Im}(B_\tau)] \sin(\theta) d\theta$$

where τ is the first time that a complex Brownian motion B exits $\mathbf{H} \setminus A$ and $\mathbf{E}_z$ denotes the expectation with respect to the law under which B starts from z.

Problem 11.

(i) Using the conformal invariance of Brownian motion, show that the hitting density (with respect to Lebesgue measure) for a complex Brownian motion starting from $z \in \mathbf{D}$ on the unit circle is given by

$$p(z, e^{i\theta}) = \frac{1}{2\pi} \frac{1-|z|^2}{|e^{i\theta} - z|^2} \quad \text{for} \quad \theta \in [0, 2\pi).$$

You may assume that the hitting density is given by the uniform distribution on $\partial \mathbf{D}$ when $z = 0$.
(ii) Suppose that u is a harmonic function on a domain $D \subseteq \mathbf{C}$. Show that for each $n \in \mathbf{N} = \{1, 2, \ldots\}$ there exists a constant $c_n > 0$ such that for all $j, k \in \mathbf{N}_0 = \{0, 1, \ldots\}$ with $j + k = n$ and $z = x + iy \in D$ we have that

$$\left| \partial_x^j \partial_y^k u(z) \right| \leq \frac{c_n}{\mathrm{dist}(z, \partial D)^n} \|u\|_\infty.$$

Problem 12. Show that there exist constants $c > 0$ and $\alpha \in (0,1)$ so that the following true. Suppose that $A \subseteq \mathbf{C}$ is a connected set which intersects both $\partial B(0,\epsilon)$ and $\partial B(0,1)$. Let B be a complex Brownian motion starting from 0 and $\tau = \inf\{t \geq 0 : |B_t| \geq 1\}$. Then

$$\mathbf{P}[B([0,\tau]) \cap A \neq \emptyset] \leq c\epsilon^\alpha.$$

[Hint: show that if $B_0 = 3/4$ then B has a positive chance of disconnecting $B(0,1/2)$ from ∞ before exiting $B(0,1) \setminus B(0,1/2)$.]

Optional problems: Riemann mapping theorem

The purpose of this sequence of problems is to prove the Riemann mapping theorem.

Optional Problem 1. Prove the Harnack inequality: suppose that u is a positive harmonic function defined on a domain D. Then for each $K \subseteq D$ compact there exists a constant $M > 0$ (independent of u) such that

$$\frac{\sup_{z\in K} u(z)}{\inf_{z\in K} u(z)} \leq M.$$

Optional Problem 2. Deduce from Problem 1 that if $f, \widetilde{f}$ are conformal transformations $D \to \mathbf{D}$ taking z to 0 and with positive derivative at z, then $f = \widetilde{f}$.

Optional Problem 3. Suppose that D is a simply connected domain with $D \neq \mathbf{C}$. Suppose that $z \in D$. Show that there exists a unique conformal transformation $f \colon D \to \mathbf{D}$ with $f(z) = 0$ and $f'(z) > 0$ using the following steps.

- Let $\mathcal{C}$ be the collection of conformal transformations f from D into a subset of $\mathbf{D}$ with $f(z) = 0$ and $f'(z) > 0$. Deduce from the Schwarz lemma that if $f \in \mathcal{C}$ then $f'(z) \leq (\mathrm{dist}(z, \partial D))^{-1}$.
- Show that $\mathcal{C}$ is non-empty.
- Suppose that (f_n) is a sequence in $\mathcal{C}$ such that, for each $K \subseteq D$ compact, we have that $f_n|_K \to f|_K$ uniformly where f is conformal on D. Show that f is either constant or injective.
- Let $M = \sup\{f'(z) : z \in \mathcal{C}\}$. Let (f_n) be a sequence of functions in $\mathcal{C}$ with $f_n'(z)$ increasing to M. Explain why there exists a subsequence (f_{n_k}) of (f_n) which converges uniformly to a map $f \colon D \to \mathbf{D}$. (Hint: use Problem 7, the Harnack inequality, and the Arzela-Ascoli theorem.) Explain why $f'(z) = M$ and deduce from the previous part that f is injective.
- Show that f is surjective onto $\mathbf{D}$. (Hint: argue by contradiction that if f is not surjective then $f'(z) < M$.)