RANDOM PLANAR GEOMETRY, LENT 2020, EXAMPLE SHEET 1

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Problem 1.

(i) Show that the cardinality of the set $\mathbf{T}_k$ of plane trees with k edges is the kth Catalan number
$$C_k = \frac{1}{k+1}\binom{2k}{k}.$$
[Hint: recall that the Catalan numbers satisfy the recursion $C_{k+1} = \sum_{i=0}^{k} C_i C_{k-i}$.]

(ii) Show that the Dyck paths of length $2k$ are in bijection with $\mathbf{T}_k$ via the contour function map.

Problem 2. Suppose that τ is a Galton-Watson tree with Geometric(1/2) offspring distribution, viewed as a plane tree. Show that the conditional law of τ given that $|\tau| = k$ is uniformly distributed on $\mathbf{T}_k$.

Problem 3. Let $p_t^*(x,y) = p_t(x,y) - p_t(x,-y)$ where $p_t(x,y)$ is the transition density for a standard Brownian motion. Show that p_t^* is the transition density for the process $B_{t\wedge\tau}$ where B is a standard Brownian motion with $B_0 > 0$ and $\tau = \inf\{t \geq 0 : B_t = 0\}$. That is, for each $0 < t_1 < \cdots < t_k$ and $x_1, \ldots, x_k > 0$ show that the law of $(B_{t_1\wedge\tau}, \ldots, B_{t_k\wedge\tau})$ has density $p_{t_1}^*(B_0, x_1) p_{t_2-t_1}^*(x_1, x_2) \cdots p_{t_k - t_{k-1}}^*(x_{k-1}, x_k)$. (The process $B_{t\wedge\tau}$ is Brownian motion killed upon hitting 0.) *[Hint: use the reflection principle.]*

Problem 4. Show that the Brownian excursion is well-defined using the following steps.

(i) The densities $\mathbf{BE}_{t_1,\ldots,t_k}$ on $\mathbf{R}_+^k$ define probability measures which are consistent. That is, show that for each $0 < t_1 < \cdots < t_{k+1} < 1$, $1 \leq j \leq k+1$ we have that
$$\mathbf{BE}_{t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{k+1}}(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_{k+1}) = \int_0^\infty \mathbf{BE}_{t_1,\ldots,t_{k+1}}(x_1,\ldots,x_{k+1}) dx_j$$
and $\int_0^\infty \cdots \int_0^\infty \mathbf{BE}_{t_1,\ldots,t_k}(x_1,\ldots,x_k) dx_1 \cdots dx_k = 1$.

(ii) There exists a unique continuous process $\mathbf{e}\colon [0,1] \to \mathbf{R}$ whose finite dimensional distributions are given by $\mathbf{BE}$. *[Hint: use the Kolmogorov-Centsov continuity criterion.]*

Explain further why $\mathbf{e}$ is Hölder-$(\frac{1}{2} - \epsilon)$ continuous for each $\epsilon > 0$.

Problem 5. Show that the tree $(\mathcal{T}_g, d_g)$ encoded by a continuous function $g\colon [0,1] \to [0,\infty)$ is an $\mathbf{R}$-tree.

Problem 6. Suppose that $\mathbf{e}$ is a Brownian excursion, let $(\mathcal{T}, d)$ be the associated CRT, and let $\pi\colon [0,1] \to \mathcal{T}$ be the associated projection map. Prove that the following statements hold a.s.

(i) The set of $t \in [0,1]$ so that $\pi(t)$ is a leaf in $\mathcal{T}$ has full Lebesgue measure. *[Hint: show that for each $t \in (0,1)$ and $\epsilon > 0$ there a.s. exists $s \in (t-\epsilon, t)$ so that $\mathbf{e}(s) < \mathbf{e}(t)$ and also $s \in (t, t+\epsilon)$ so that $\mathbf{e}(s) < \mathbf{e}(t)$.]*

(ii) Every $a \in \mathcal{T}$ has multiplicity at most 3. *[Hint: Show that the set of local minima of $\mathbf{e}$ is countable and distinct.]*

(iii) The set of $a \in \mathcal{T}$ with multiplicity 3 is countable.

Problem 7.

(i) Prove that every compact metric space can be isometrically embedded into ℓ_∞ (the space of bounded real sequences equipped with the metric $d((a_n),(b_n)) = \sup_n |a_n - b_n|$). *[Hint: let (x_n) be a countable dense subset of (X,d) and consider the map $X \to \ell_\infty$ defined by $x \mapsto (d(x,x_n))_{n=1}^\infty$. Check that this map defines an isometry on (x_n) hence extends to an isometry on X.]*

(ii) Deduce the triangle inequality for the Gromov-Hausdorff distance.

Problem 8. Suppose that (X,d), (X',d') are compact metric spaces. Show that

$$d_{\mathrm{GH}}(X,X') = \frac{1}{2}\inf_{\mathcal{R}} \mathrm{dis}(\mathcal{R})$$

where the infimum is over all correspondences $\mathcal{R}$ in $X \times X'$ and

$$\mathrm{dis}(\mathcal{R}) = \sup\{|d(x,y) - d'(x',y')| : (x,x'),(y,y') \in \mathcal{R}\}$$

is the distortion of $\mathcal{R}$ using the following steps.

(i) Show that $d_{\mathrm{GH}}(X,X') = \inf\{D_{\mathrm{H}}(X,X') : D$ is a metric on $X \coprod X'$ with $D|_X = d,\ D|_{X'} = d'\}$ where $X \coprod X'$ denotes the disjoint union of X and X'.

(ii) Deduce that if $d_{\mathrm{GH}}(X,X') < \epsilon$ then $\mathcal{R} = \{(x,x') : D(x,x') < \epsilon\}$ where D is a metric on $X \coprod X'$ as above defines a correspondence with $\mathrm{dis}(\mathcal{R}) < 2\epsilon$. Conclude that $\frac{1}{2}\mathrm{dis}(\mathcal{R}) \leq d_{\mathrm{GH}}(X,X')$.

(iii) Also show that if $\mathrm{dis}(\mathcal{R}) < 2\epsilon$ then $D|_X = d$, $D|_{X'} = d'$, and

$$D(x,x') = \inf\{d(x,y) + d'(x',y') + \epsilon : (y,y') \in \mathcal{R}\} \quad \text{for} \quad x \in X, \quad x' \in X'$$

defines a metric on $X \coprod X'$ with $d_{\mathrm{H}}(X,X') < \epsilon$. Conclude that $d_{\mathrm{GH}}(X,X') \leq \frac{1}{2}\mathrm{dis}(\mathcal{R})$.

Problem 9. Prove the following version of the local central limit theorem using Stirling's formula. Suppose that $S(n) = \sum_{j=1}^n \xi_j$ where the (ξ_n) are i.i.d. with $\mathbf{P}[\xi_1 = 1] = \mathbf{P}[\xi_1 = -1] = 1/2$. Using Stirling's formula, prove that for every $\epsilon > 0$

$$\lim_{n\to\infty} \sup_{x\in\mathbf{R}} \sup_{s\geq\epsilon} \left|\sqrt{n}\mathbf{P}\left[S(\lfloor ns \rfloor) = \lfloor x\sqrt{n}\rfloor \text{ or } \lfloor x\sqrt{n}\rfloor + 1\right] - 2p_s(0,x)\right| = 0$$

where $p_s(x,y)$ is the transition kernel for Brownian motion.

Problem 10. Suppose that C_k is the contour function for τ chosen uniformly at random from $\mathbf{T}_k$. Show that the family of functions $[0,1] \to \mathbf{R}_+$ defined by $t \mapsto C_k(2kt)/\sqrt{2k}$ is tight using the following steps. Suppose that $i,j \in \{0,\ldots,2k\}$ with $j > i$.

(i) Explain why $|C_k(j) - C_k(i)| \leq C_k(j) + C_k(i) - 2\min_{i\leq\ell\leq j} C_k(\ell)$

(ii) Explain why $C_k(j) + C_k(i) - 2\min_{i\leq\ell\leq j} C_k(\ell) \overset{d}{=} C_k(j-i)$ *[Hint: re-root τ so that the ith vertex in the contour exploration becomes the root.]*

(iii) Show that for each $p > 1$ there exists a constant $c_p > 0$ so that $\mathbf{E}[(C_k(i))^{2p}] \leq c_p i^p$. *[Hint: use the formula for $\mathbf{P}[C_k(i) = x]$ derived in the proof of the convergence of the first order marginal.]*

(iv) Conclude that there exists a constant $c_p > 0$ so that for each $0 \leq s < t \leq 1$ we have that $\mathbf{E}[|(C_k(2kt) - C_k(2ks))/\sqrt{2k}|^{2p}] \leq c_p(t-s)^p$.

Problem 11. Prove *Euler's formula.* That is, show that if m is a map then

$$\#\mathbf{V}(m) - \#\mathbf{E}(m) + \#\mathbf{F}(m) = 2.$$

[Hint: consider how $\mathbf{V}(m)$, $\mathbf{E}(m)$, and $\mathbf{F}(m)$ change when removing an edge.]

Deduce that m is a quadrangulation with n faces then $\#V(m) = n + 2$.

Problem 12. Prove that q if is a quadrangulation then it is bipartite.