STOCHASTIC CALCULUS, LENT 2016, EXAMPLE SHEET 4

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Problem 1. Let b be bounded and measurable. Use Girsanov's theorem to construct a weak solution to the SDE

$$dX_t = b(X_t)dt + dW_t$$

over the finite (non-random) time interval $[0, T]$.

Problem 2. Show that the SDE

$$dX_t = 3X_t^{1/3}dt + 3X_t^{2/3}dB_t, \quad X_0 = 0$$

has strong existence but not pathwise uniqueness.

Problem 3. Find the unique strong solution to the SDE

$$dX_t = \frac{1}{2}X_t dt + \sqrt{1 + X_t^2}dB_t, \quad X_0 = x.$$

(Hint: consider the change of variables $Y_t = \sinh^{-1}(X_t)$.)

Problem 4. Construct a filtered probability space on which a Brownian motion B and an adapted process X are defined and such that

$$dX_t = \frac{X_t}{t}dt + dB_t, \quad X_0 = 0.$$

Is X adapted to the filtration generated by B? Is B a Brownian motion in the filtration generated by X?

Problem 5. Let X be a solution of the SDE

$$dX_t = X_t g(X_t)dB_t$$

where g is bounded and $X_0 = x > 0$ is non-random.

(1) Show that $\mathbb{P}[X_t > 0 \quad \text{for all} \quad t \geq 0] = 1$. Hint: apply Ito's formula to

$$X_t \exp\left(-\int_0^t g(X_s)dB_s + \frac{1}{2}\int_0^t g^2(X_s)ds\right).$$

(2) Show that $\mathbb{E}[X_t] = X_0$ for all $t \geq 0$.

(3) Fix a non-random time horizon $T > 0$. Show that there exists a measure $\widehat{\mathbb{P}}$ on $(\Omega, \mathcal{F}_T)$ which is mutually absolutely continuous with respect to $\mathbb{P}$ and a $\widehat{\mathbb{P}}$-Brownian motion $\widehat{B}$ such that

$$dY_t = Y_t g(1/Y_t)d\widehat{B}_t$$

where $Y_t = 1/X_t$.

Problem 6. Establish the following generalized version of the Feynman-Kac formula. Suppose that v satisfies

$$-\frac{\partial v}{\partial t} + kv = \frac{1}{2}\Delta v + g, \quad \text{on} \quad [0,T) \times \mathbb{R}^d$$

where $v(T,x) = f(x)$ for $x \in \mathbb{R}^d$. Let B be a standard Brownian motion and show that v admits the stochastic representation

$$\begin{aligned} v(t,x) =& \mathbb{E}_x \Bigg[f(B_{T-t}) \exp\left(- \int_0^{T-t} k(B_s) ds \right) \\ &+ \int_0^{T-t} g(t+u, B_u) \exp\left(- \int_0^u k(B_s) ds \right) du \Bigg] \end{aligned}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^d$.

Problem 7. Consider the Cauchy problem for the "quasi-linear" parabolic equation

$$\frac{\partial V}{\partial t} = \frac{1}{2}\Delta V - \frac{1}{2}\|\nabla V\|^2 + k \quad \text{on} \quad (0,\infty) \times \mathbb{R}^d,$$

with $V(0,x) = 0$ for $x \in \mathbb{R}^d$ where $k \colon \mathbb{R}^d \to [0,\infty)$ is a continuous function. Show that the only solution $V \colon [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ which is continuous on its domain, of class $C^{1,2}$ on $(0,\infty) \times \mathbb{R}^d$, and satisfies the quadratic growth condition for every $T > 0$:

$$-V(t,x) \le C + a\|x\|^2, \quad (t,x) \in [0,T] \times \mathbb{R}^d$$

where $T > 0$ is arbitrary and $0 < a < 1/(2Td)$ is given by

$$V(t,x) = -\log \mathbb{E}_x \left[\exp\left(- \int_0^t k(W_s) ds \right) \right]$$

for $t \ge 0$ and $x \in \mathbb{R}^d$.

Problem 8. Suppose that $b \colon \mathbb{R}^d \to \mathbb{R}$ and $\sigma \colon \mathbb{R}^d \to M^{d\times d}(\mathbb{R})$ are bounded and continuous. For each n, j, we let $t_j^n = n2^{-j}$ and we let $\psi_n(t) = t_j^n$ if $t \in [t_j^n, t_{j+1}^n)$. Assume that X^n solves

$$X_t^n = X_0^n + \int_0^t b(X_{\psi_n(u)}^n) du + \int_0^t \sigma(X_{\psi_n(u)}^n) dB_u.$$

Show that for each $m, T > 0$ there exist a constant $C > 0$ such that

$$\mathbb{E}[\|X_t^n - X_s^n\|^{2m}] \le C(t-s)^m \quad \text{for all} \quad 0 \le s < t \le T. \tag{0.1}$$

Explain what it means for the sequence (X^n) to be tight in the space $C([0,T], \mathbb{R}^d)$ and explain why (0.1) implies that (X^n) is tight. (Hint: look at the proof of Kolmogorov's continuity criterion.)

Problem 9. Consider the SDE

$$dX_t = X_t^2 dB_t.$$

(1) By considering the process $\widetilde{X}_t = 1/\|B_t - \xi\|$ where B is a three-dimensional Brownian motion and ξ is a standard Gaussian in $\mathbb{R}^3$ independent of B, show that the SDE has a weak solution.

(2) Verify that both

$$u^1(t,x) = x\left(2\Phi(1/(x\sqrt{t})) - 1\right) \quad \text{and} \quad u^2(x,t) = x$$

solve the PDE

$$\frac{\partial u}{\partial t} = \frac{x^4}{2} \cdot \frac{\partial^2 u}{\partial x^2}, \quad u(0,x) = x.$$

(3) Which of these solutions corresponds to $u(t,x) = \mathbb{E}_x[X_t]$?