Random Surfaces and Quantum Loewner Evolution

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Overview

Part I: Picking surfaces at random

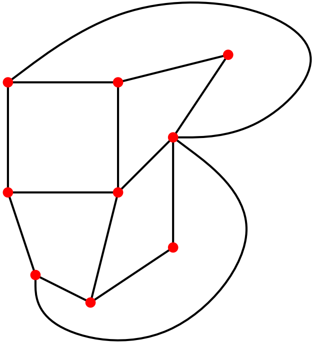
1. Discrete: random planar maps
2. Continuum: Liouville quantum gravity
3. Conjectured relationship

Part II: Quantum Loewner evolution

1. New universal family of growth processes
2. Tool to relate random planar maps to Liouville quantum gravity
3. Connected to many different topics in probability:
   RPM, TBM, LQG, GFF, SLE, DLA, FPP, DBM, KPZ, KPZ

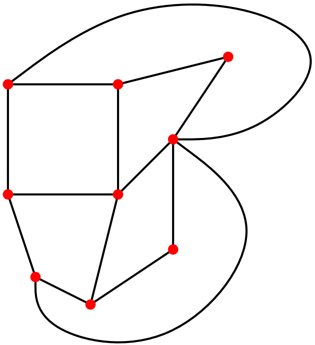
Part I: Picking surfaces at random

Random planar maps



- A **planar map** is a finite graph embedded in the plane

Random planar maps



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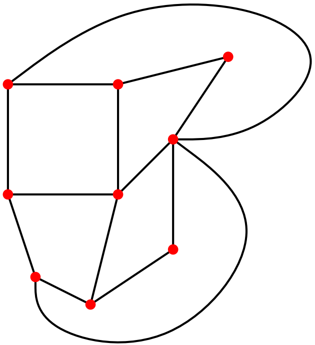

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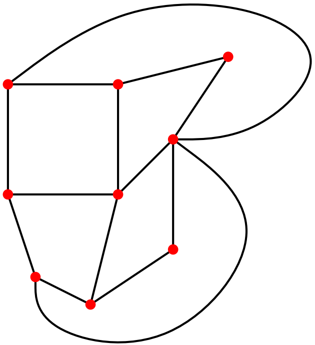

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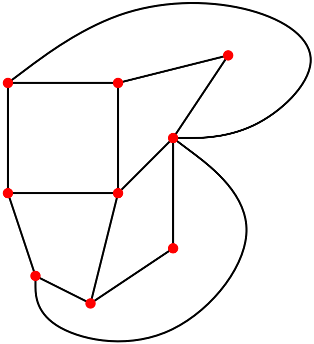

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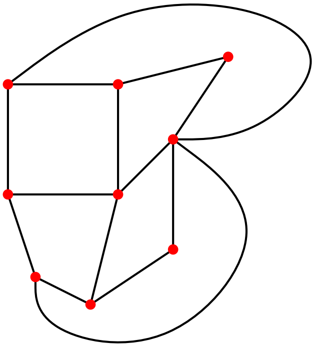

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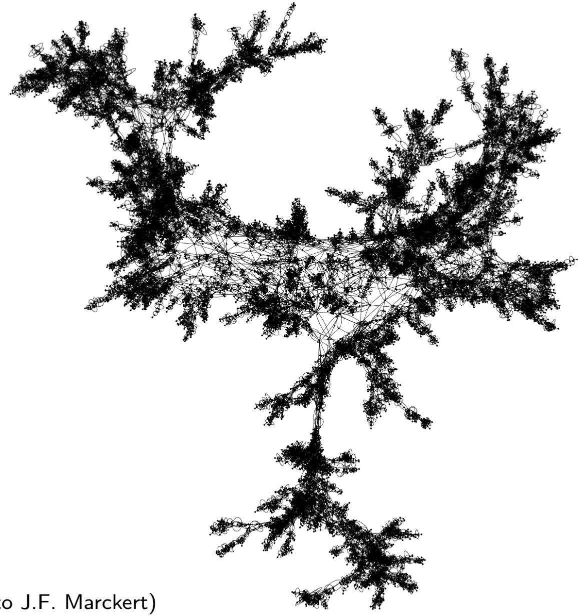
- In this talk, interested in **uniformly random quadrangulations** — **random planar map** (RPM).
- First studied by Tutte in 1960s while working on the four color theorem
  - **Combinatorics**: enumeration formulas
  - **Physics**: statistical physics models: percolation, Ising, UST ...
  - **Probability**: "uniformly random surface," Brownian surface

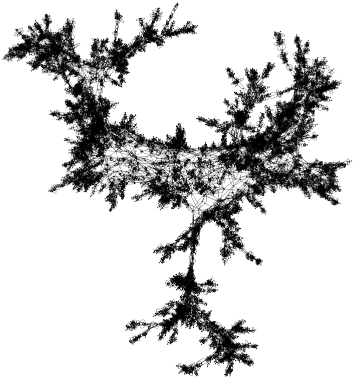
Random quadrangulation with 25,000 faces



(Simulation due to J.F. Marckert)

Structure of large random planar maps

- RPM as a **metric space**. Is there a limit?



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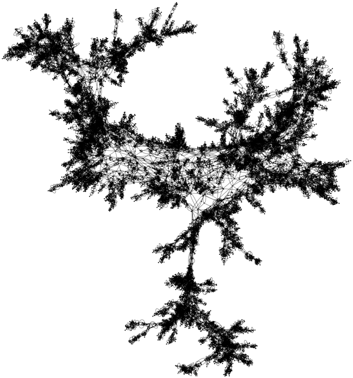

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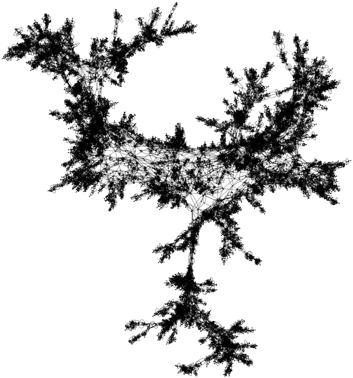

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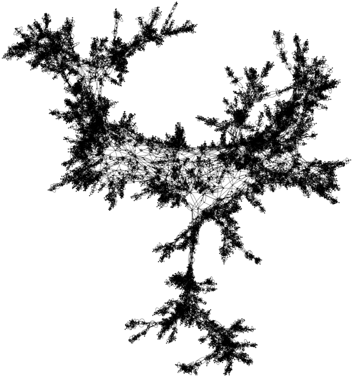

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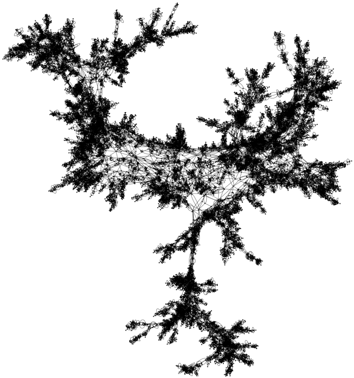

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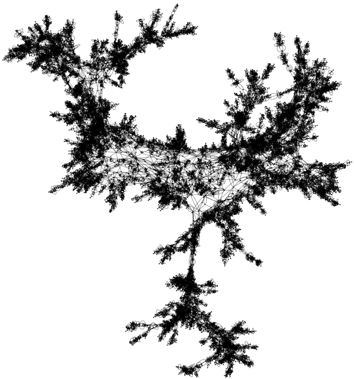

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Important tool: bijections which encode the surface using a gluing of a pair of trees

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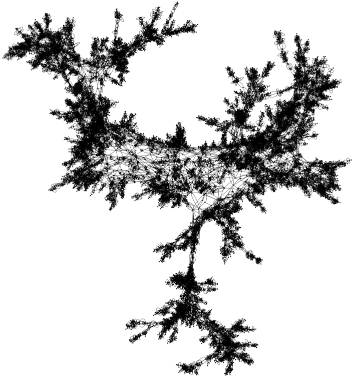

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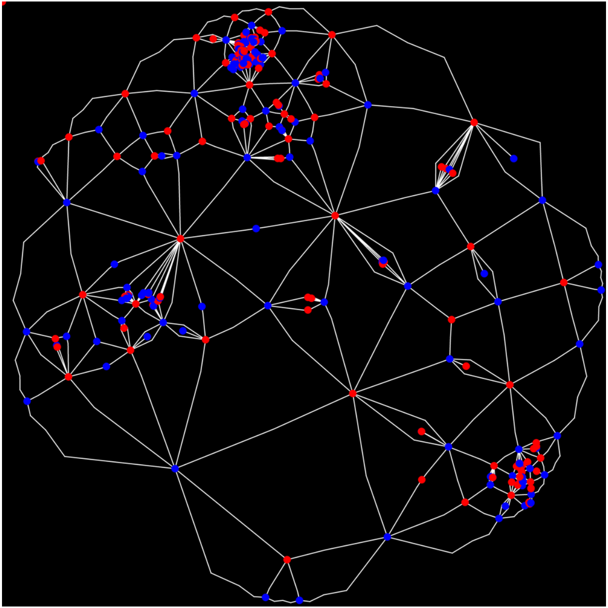
Brownian map also described in terms of trees (CRT)

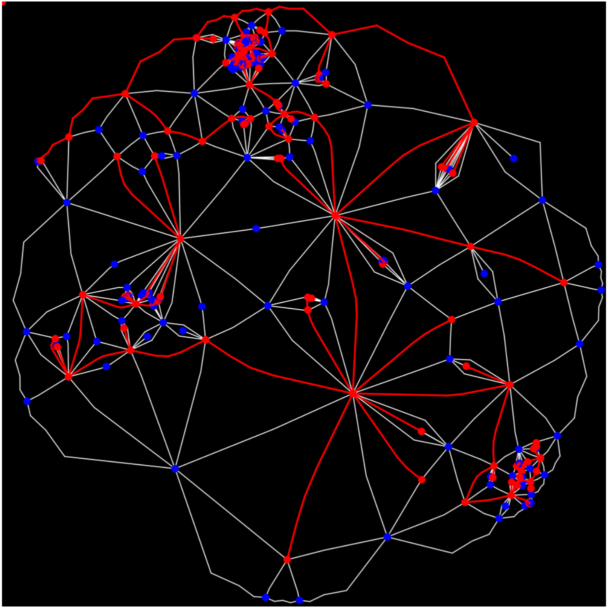
(Markert-Mokkadem)

Random quadrangulation



Sampled using Sheffield's H-C bijection.

Red tree



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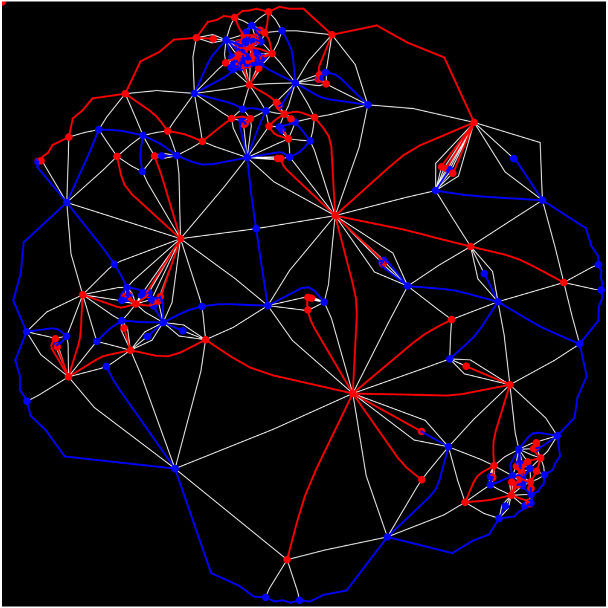
Red and blue trees



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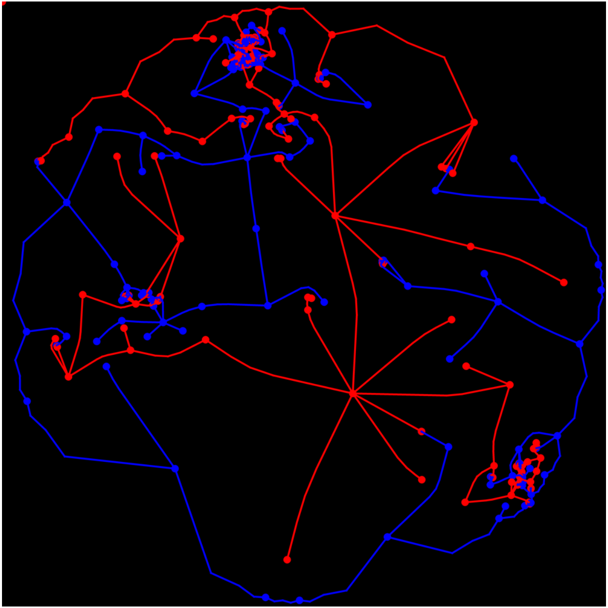
Red and blue trees alone do not determine the map structure



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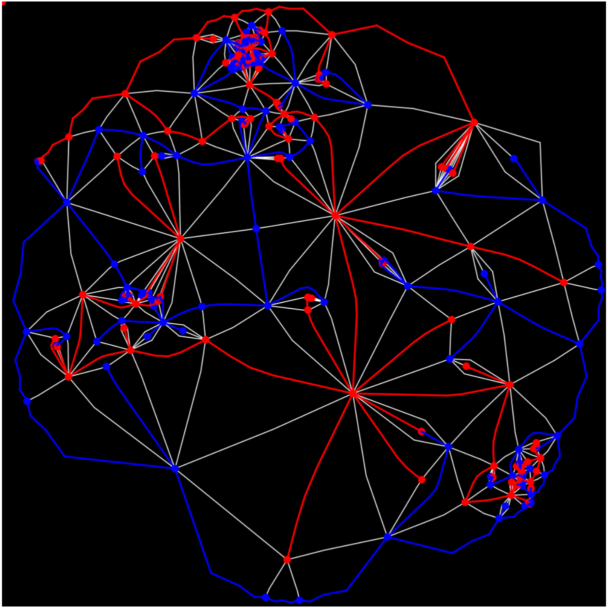
Random quadrangulation with red and blue trees



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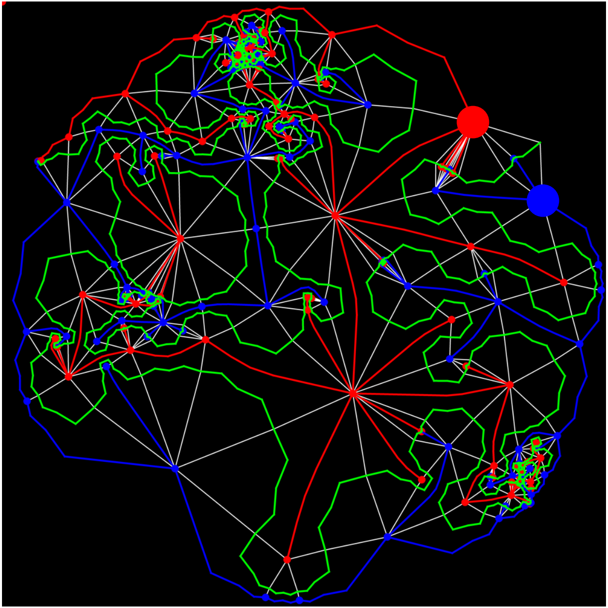
Path snaking between the trees. Encodes the trees and how they are glued together.



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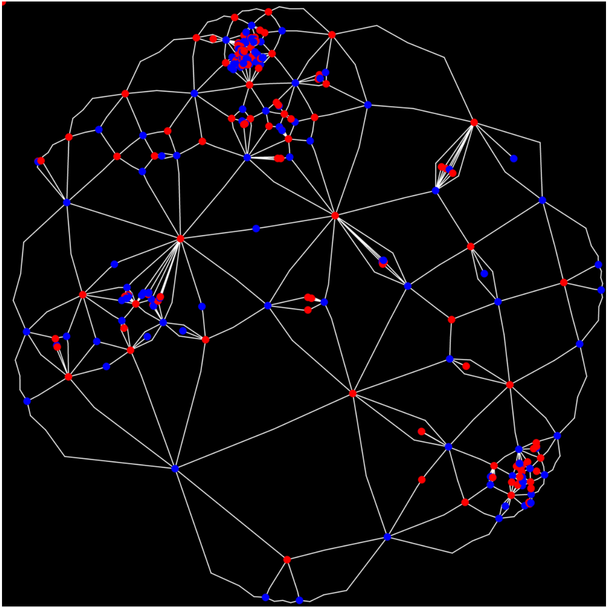
How was the graph embedded into $\mathbf{R}^2$?



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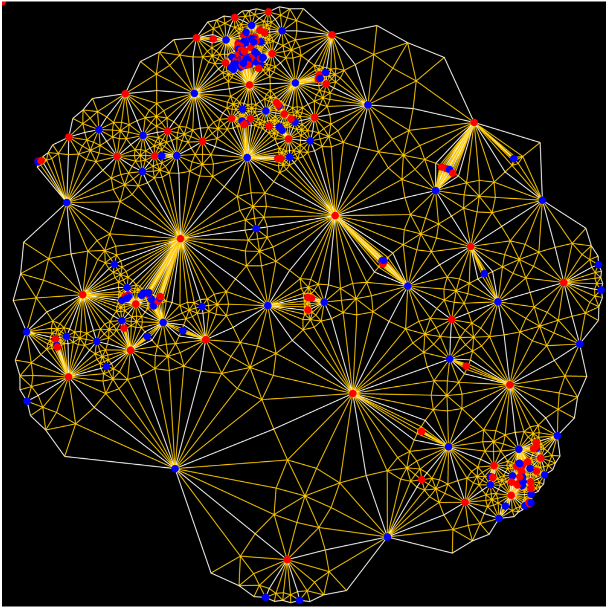
Can subivide each quadrilateral to obtain a triangulation without multiple edges.



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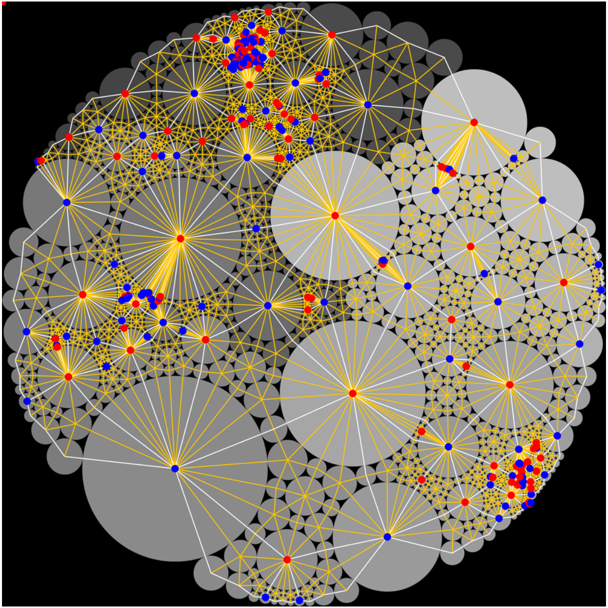
Circle pack the resulting triangulation.



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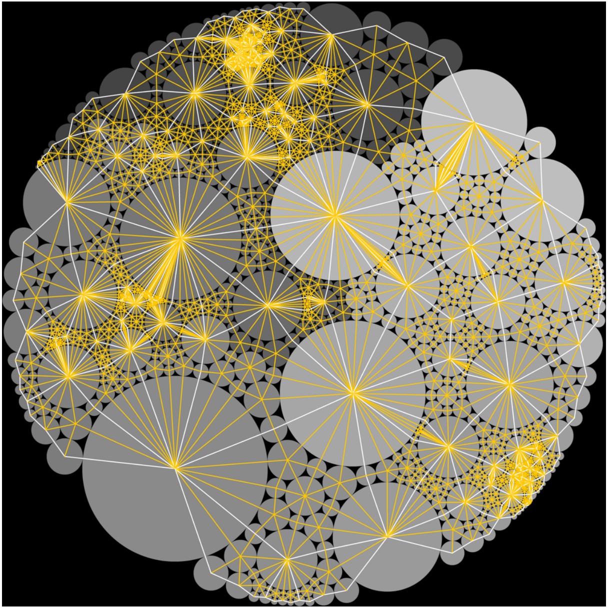

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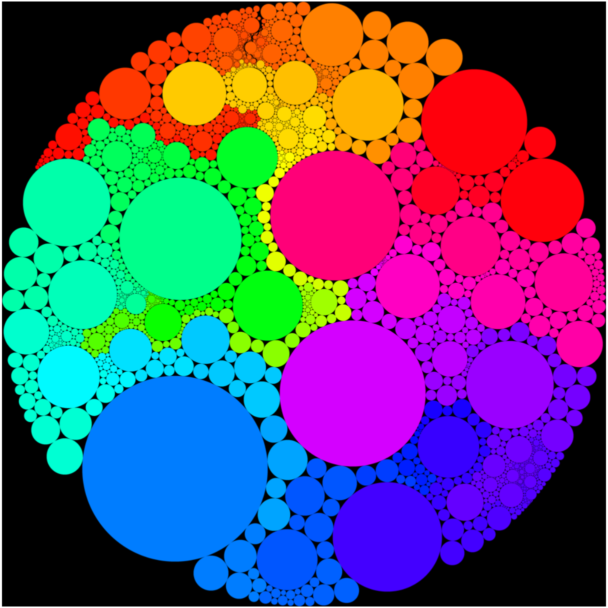

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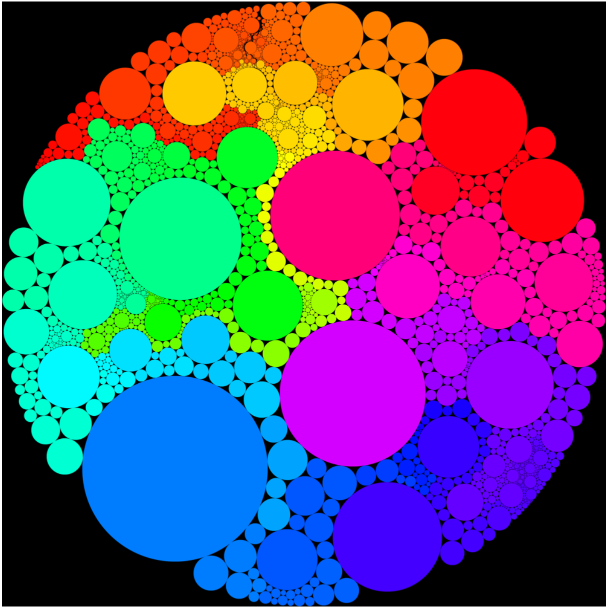
What is the "limit" of this embedding? Circle packings are related to conformal maps.



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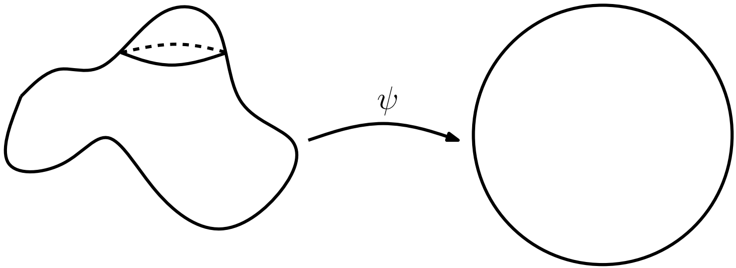
Picking a surface at random in the continuum

Uniformization theorem: every simply connected Riemannian surface can be conformally mapped to either the unit disk, the plane, or the sphere $\mathbf{S}^2$ in $\mathbf{R}^3$



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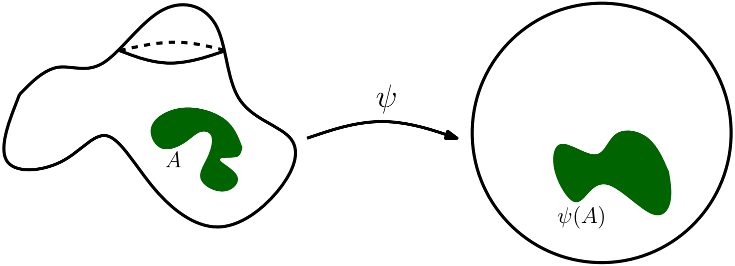

Isothermal coordinates: Metric for the surface takes the form $e^{\rho(z)}dz$ for some smooth function ρ where dz is the Euclidean metric.

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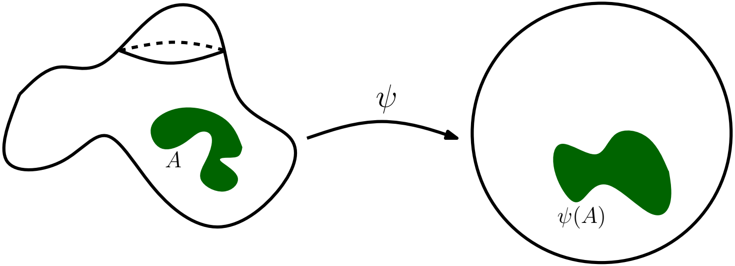

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$\Rightarrow$ Can parameterize the space of surfaces with smooth functions.

- If $\rho = 0$, get the same surface
- If $\Delta\rho = 0$, i.e. if ρ is harmonic, the surface described is flat

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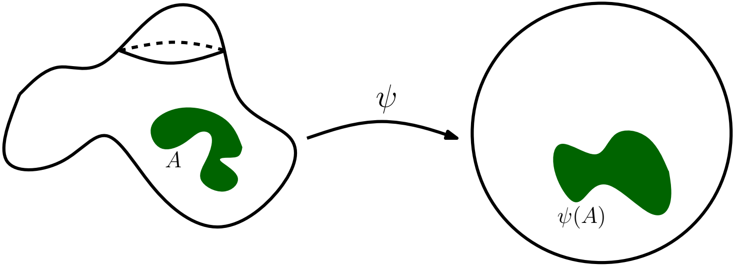

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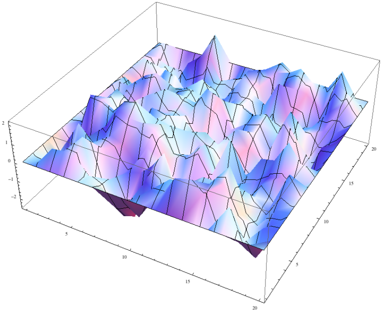
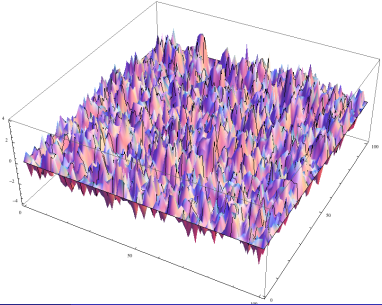
Question: Which measure on ρ? If we want our surface to be a perturbation of a flat metric, natural to choose ρ as the canonical perturbation of a harmonic function.

The Gaussian free field

- The **discrete Gaussian free field** (DGFF) is a **Gaussian random surface** model.





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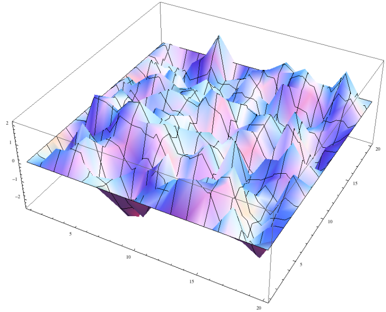
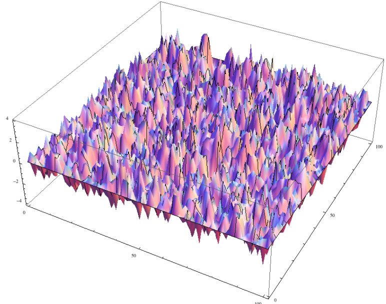
- Gaussian measure on functions $h\colon D \to \mathbf{R}$ for $D \subseteq \mathbf{Z}^2$ and $h|_{\partial D} = \psi$ where
  - **Covariance**: Green's function for SRW
  - **Mean Height**: harmonic extension of ψ





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- Density with respect to Lebesgue measure on $\mathbf{R}^{|D|}$:

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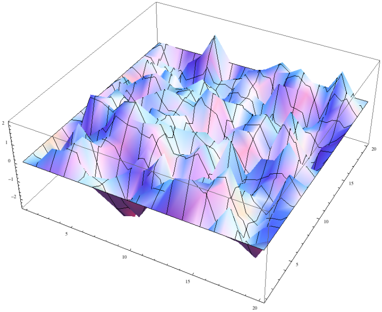

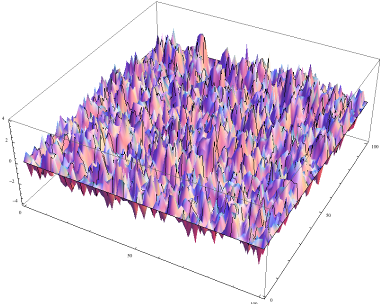

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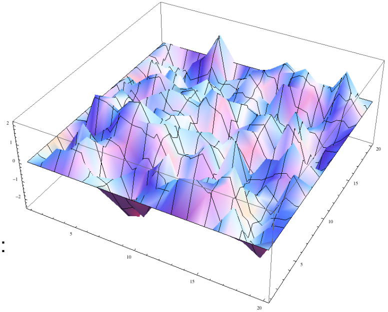

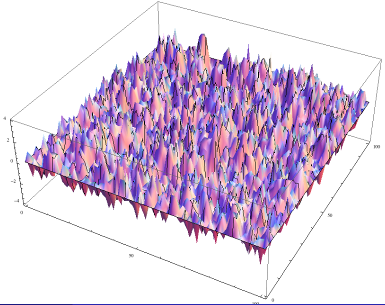

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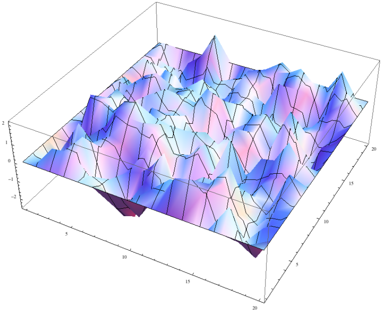
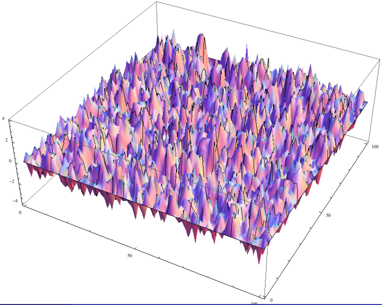
- Natural perturbation of a harmonic function
- Fine mesh limit: converges to the continuum GFF, i.e. the standard Gaussian wrt the **Dirichlet inner product**

$$(f,g)_\nabla = \frac{1}{2\pi} \int \nabla f(x) \cdot \nabla g(x) dx.$$





Liouville quantum gravity

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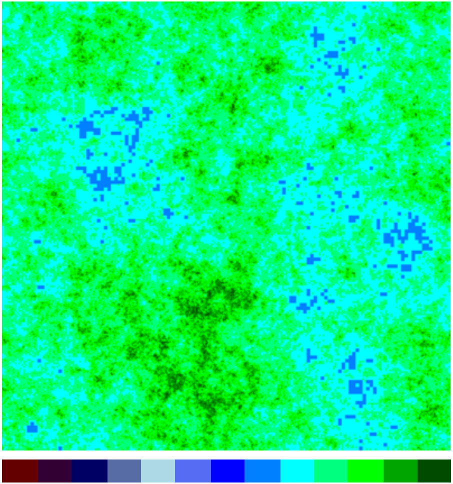

(Number of subdivisions)

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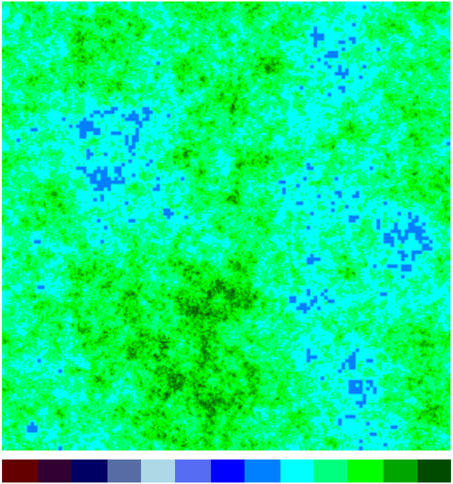

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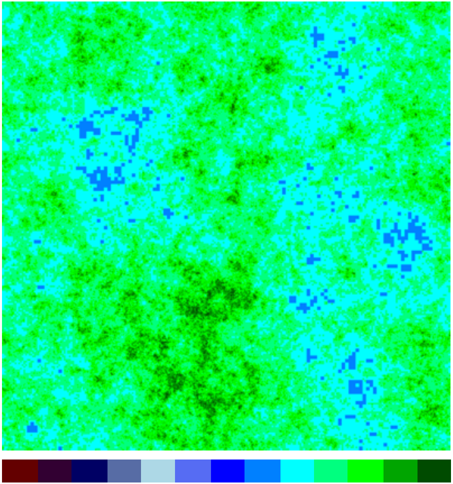

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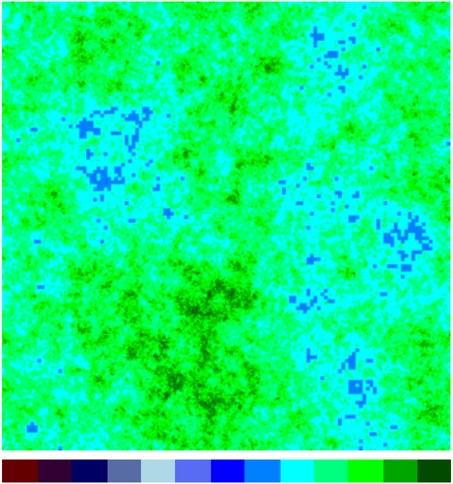

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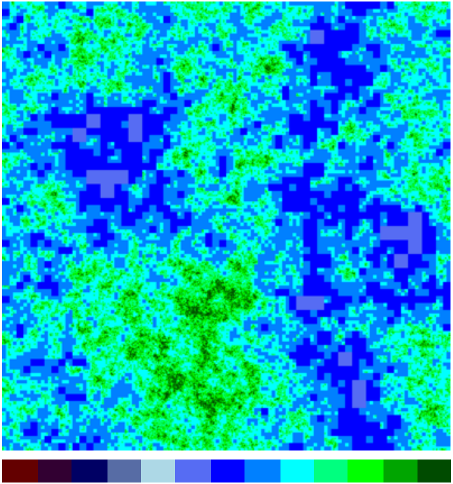
$\gamma = 1.0$



(Number of subdivisions)

Liouville quantum gravity

- Liouville quantum gravity: $e^{\gamma h(z)} dz$ where h is a GFF and $\gamma \in [0,2)$
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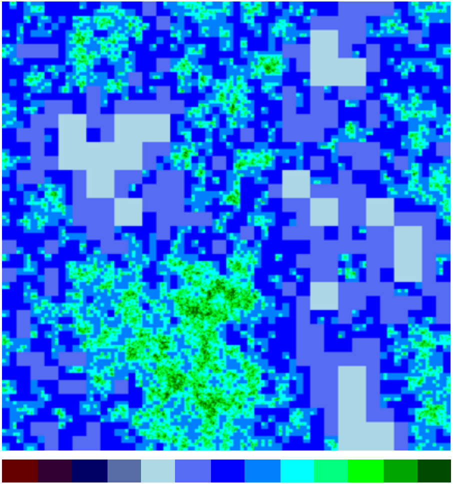
$\gamma = 1.5$



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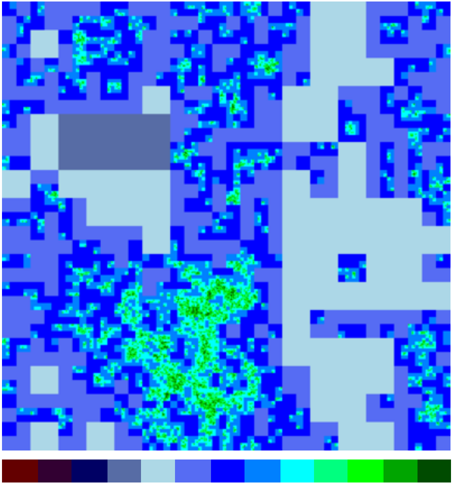

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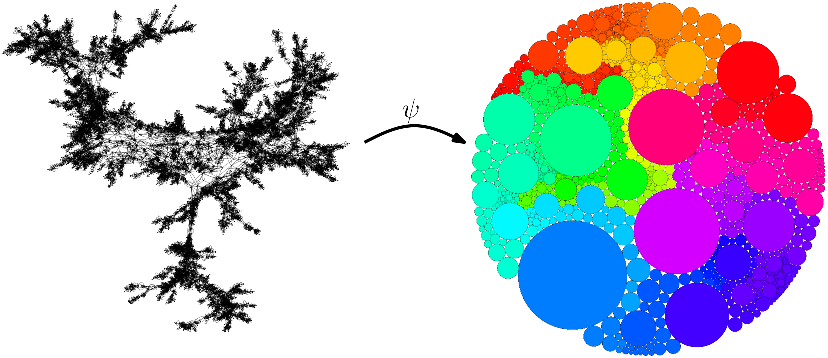
Conjecture: $\sqrt{8/3}$-LQG = TBM



(Simulation due to J.-F. Marckert)

1. **Measures:** show that the conformally mapped discrete area measures converge to LQG area measure

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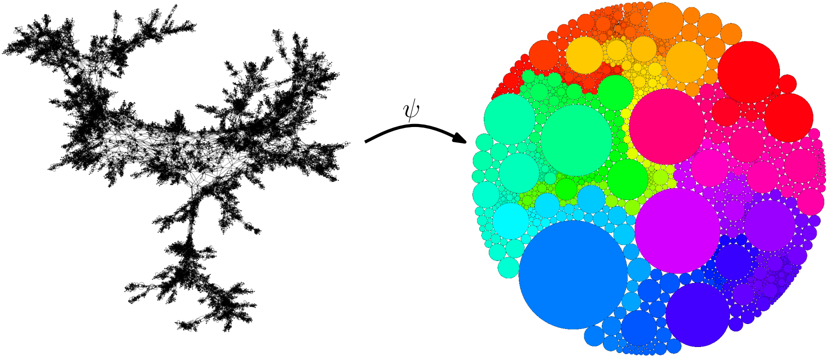

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1. **Measures:** show that the conformally mapped discrete area measures converge to LQG area measure
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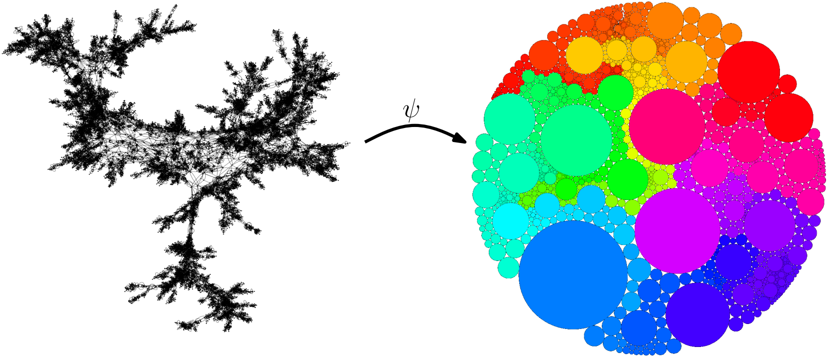

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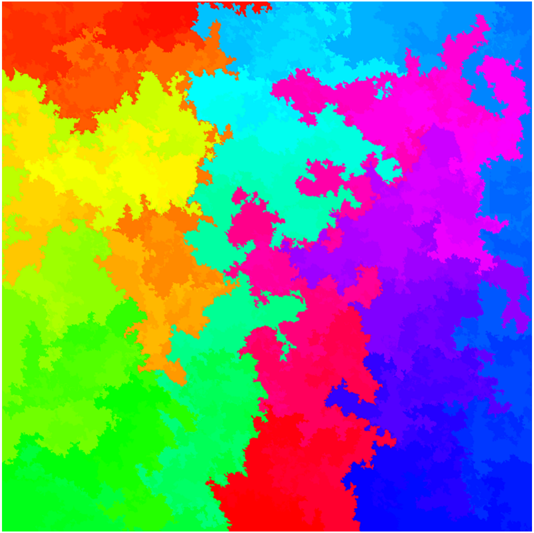
1. **Measures:** show that the conformally mapped discrete area measures converge to LQG area measure
2. **Coding functions:** put a space-filling path and coding function on LQG and show that it is the limit of the coding functions for the RPMs
3. **Metric spaces:** put a metric on LQG and show that it is isometric to TBM, the metric space limit of RPMs

Continuum space-filling path



Space-filling SLE_6 on a LQG surface. Random path which encodes the limit of a RPM.

Recap

Two natural ways to pick surfaces at random

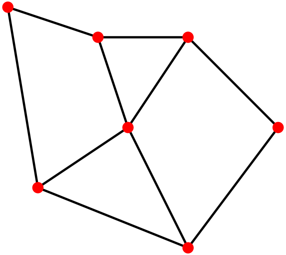
- **Discrete:** random planar maps
- **Continuum:** Liouville quantum gravity $e^{\gamma h(z)} dz$, h a GFF
- Conjectured to be the same for $\gamma = \sqrt{8/3}$
- LQG only made sense of so far as a **measure space**

Next part: describe new growth process which can be used to endow $\sqrt{8/3}$-LQG with a metric space structure

Part II:
Quantum Loewner Evolution

Detour: first passage percolation (FPP)

- Associate with a graph (V, E) i.i.d. $\exp(1)$ edge weights



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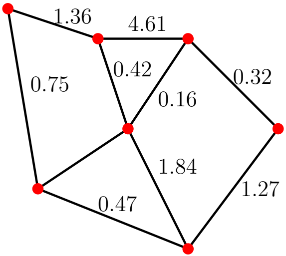

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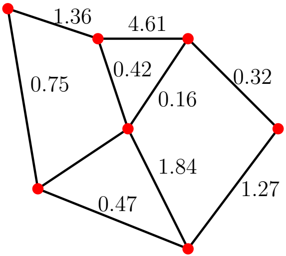

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- Associate with a graph (V, E) i.i.d. exp(1) edge weights
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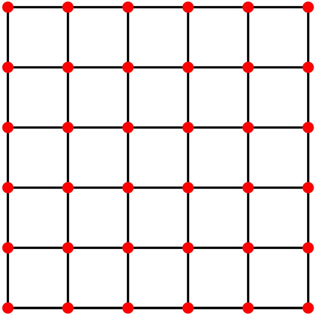
- On $\mathbf{Z}^2$?



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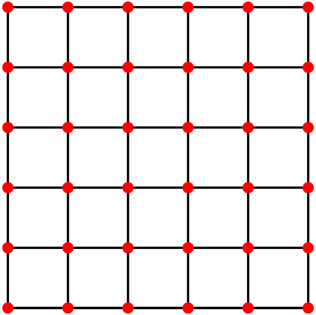

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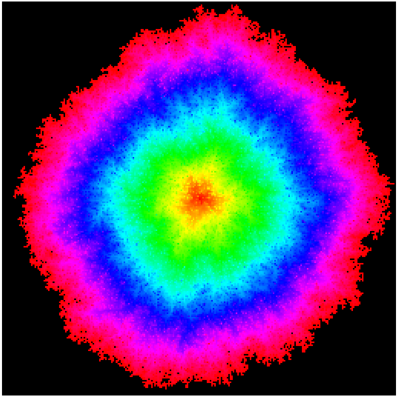

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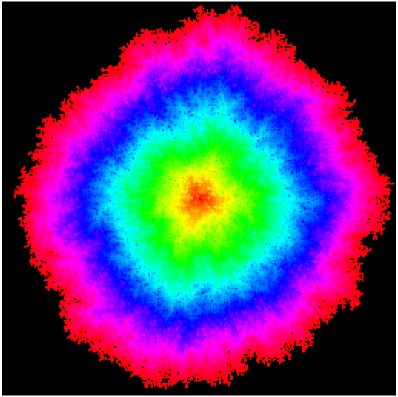

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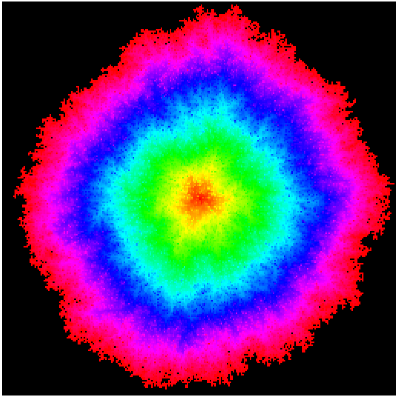

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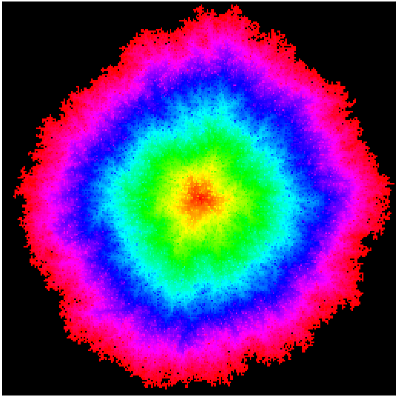

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- Computer simulations show that it is not a Euclidean disk
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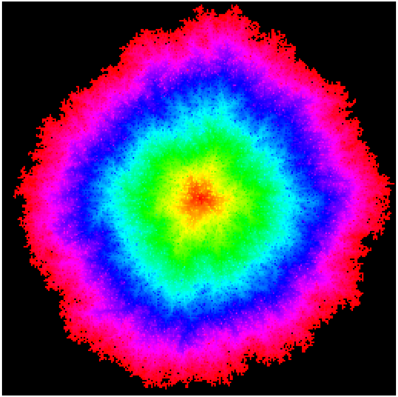
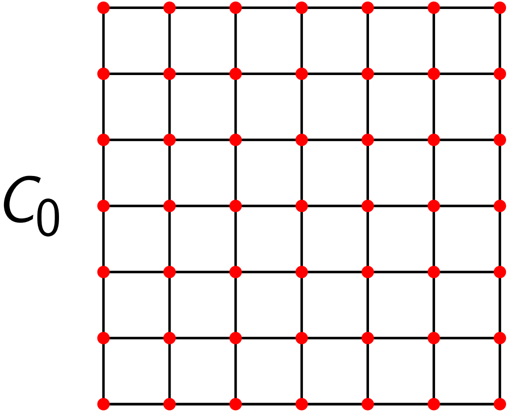
- Vahidi-Asl and Weirmann (1990) showed that the rescaled ball converges to a disk if $\mathbf{Z}^2$ is replaced by the Voronoi tesselation associated with a Poisson process



Markovian formulation

Rather than sampling all of the edge weights at once, can explore the FPP metric ball starting from a point in a Markovian way.



Due to the memoryless property of the exponential distribution, can sample the cluster C_{n+1} from C_n by selecting an edge uniformly at random on ∂C_n, and then adding the vertex which is attached to it.

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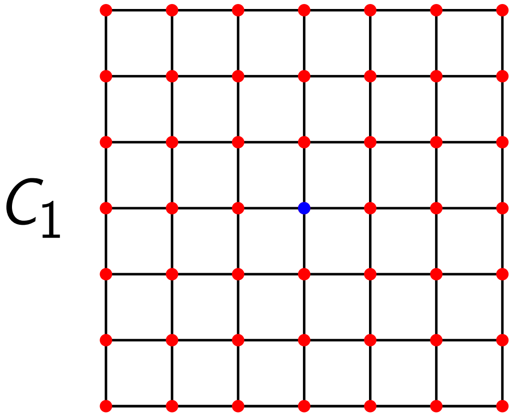

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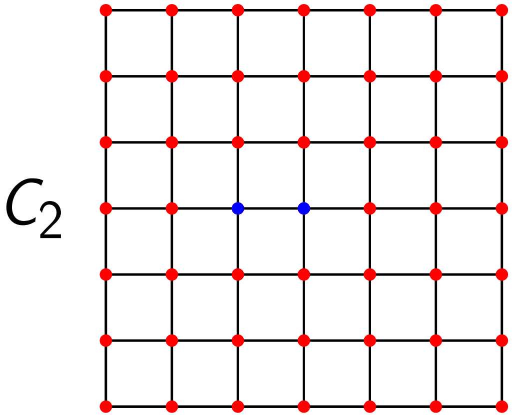

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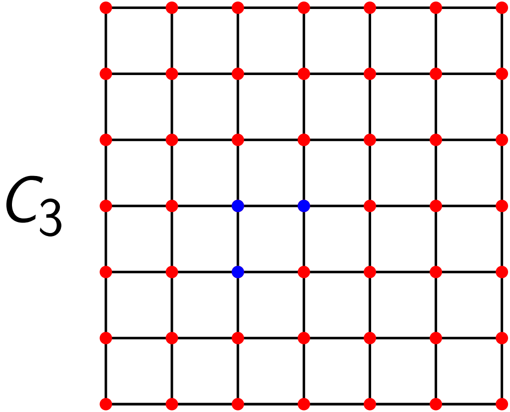

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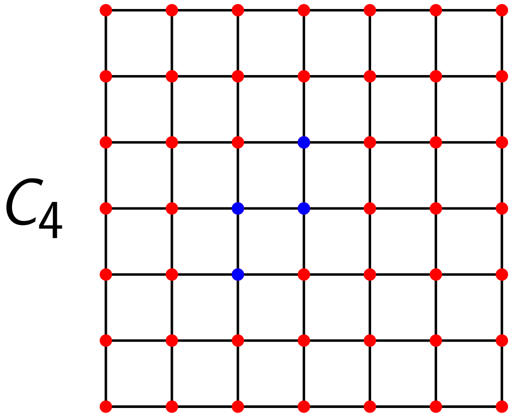

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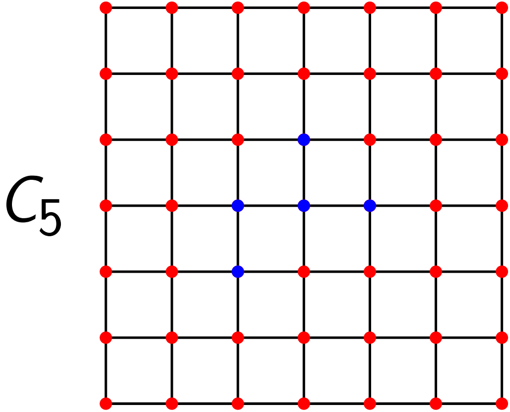

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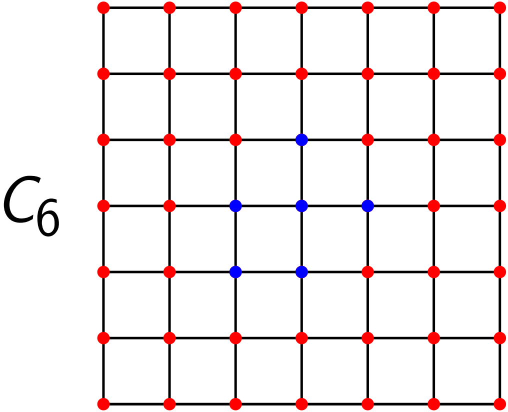

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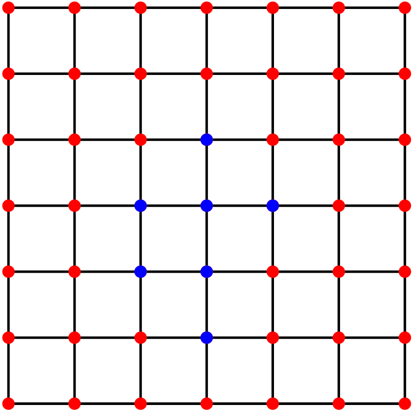
C_7



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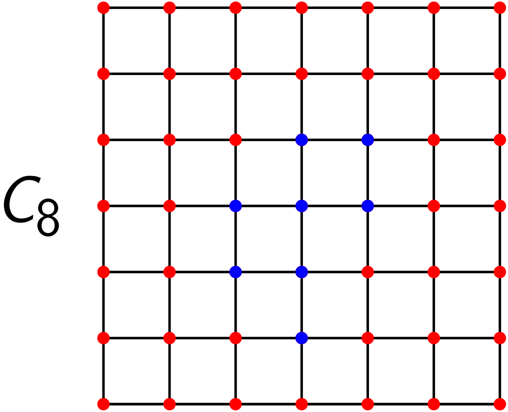

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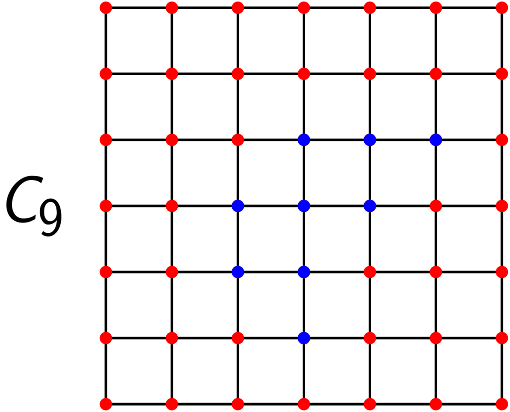

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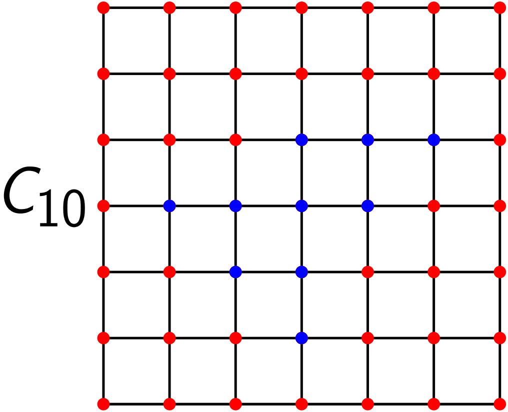

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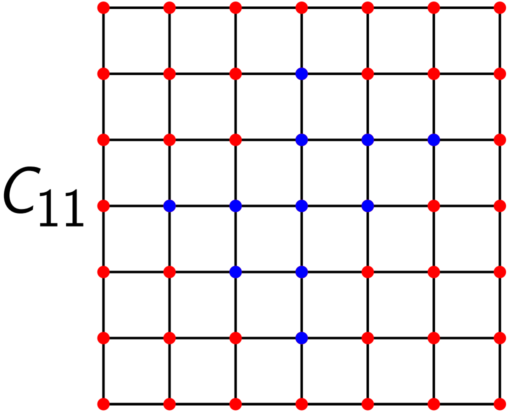

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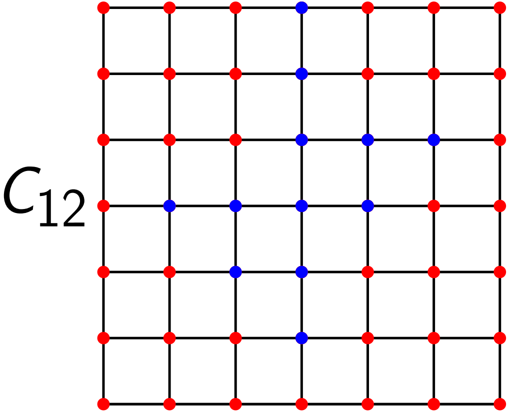

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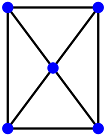
First passage percolation on random planar maps I

- Random planar map, random vertex x. Perform FPP from x.



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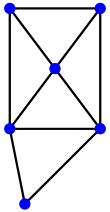

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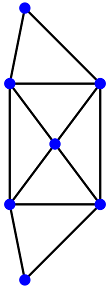

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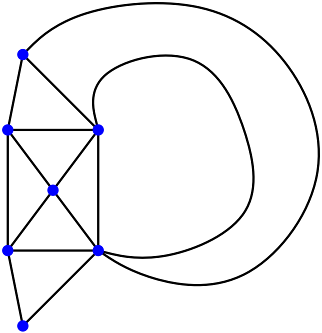

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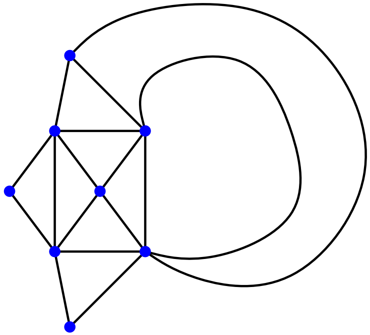

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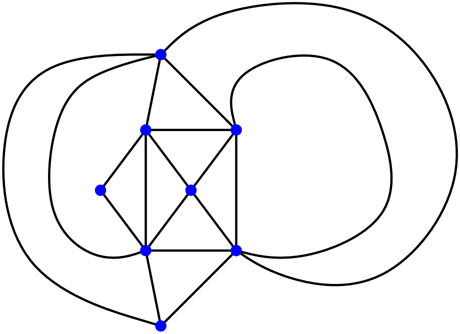

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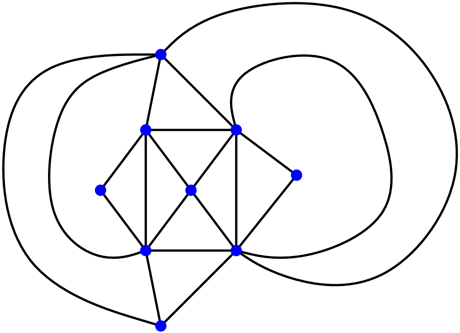

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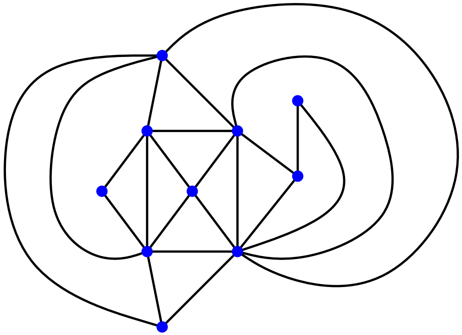

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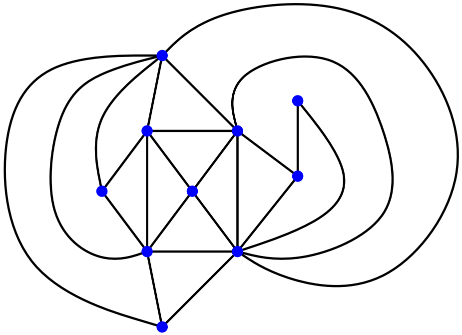

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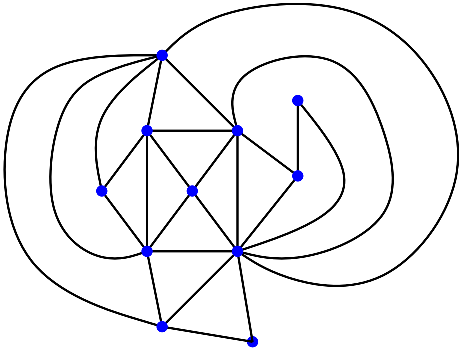

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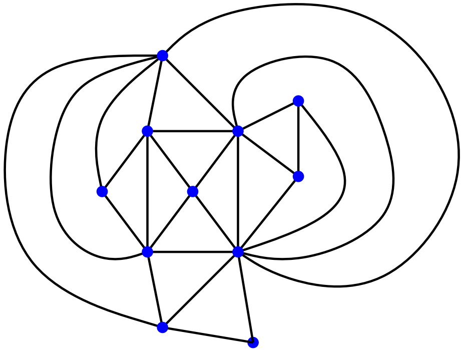

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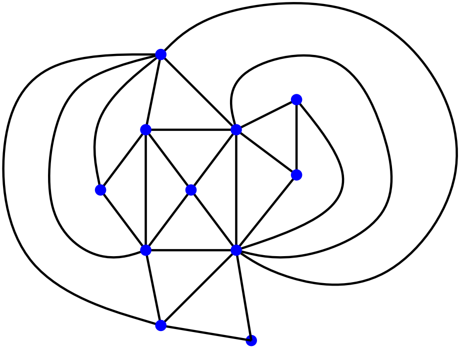

Important observations:

- Conditional law of map given ball at time n only depends on the boundary lengths of the outside components.

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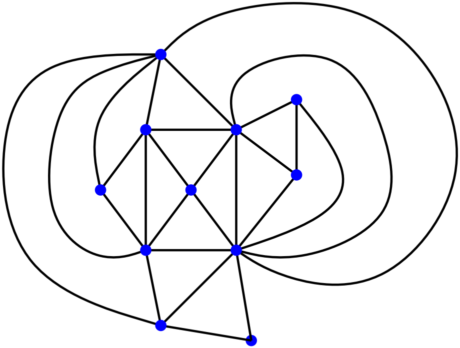

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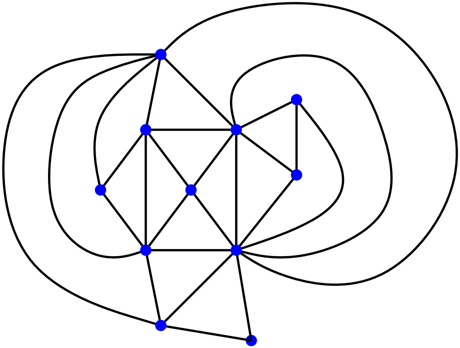

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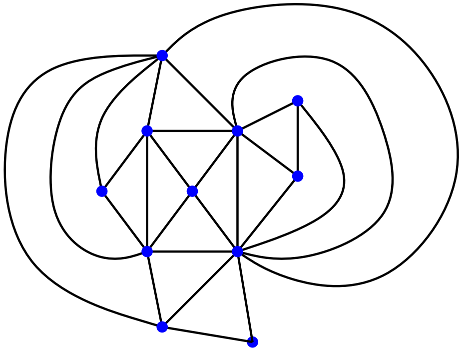

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Belief: Isotropic enough so that at large scales this is close to a ball in the graph metric

First passage percolation on random planar maps II

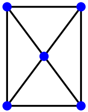
Goal: Make sense of FPP in the continuum on top of a LQG surface

- We do not know how to take a continuum limit of FPP on a random planar map and couple it directly with LQG
- Explain a discrete variant of FPP that involves two operations that we do know how to perform in the continuum:
  - Sample random points according to boundary length
  - Draw (scaling limits of) critical percolation interfaces (SLE_6)

First passage percolation on random planar maps III

Variant:

- Pick two edges on outer boundary of cluster



First passage percolation on random planar maps III

Variant:

- Pick two edges on outer boundary of cluster
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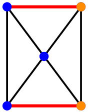

First passage percolation on random planar maps III

Variant:

- Pick two edges on outer boundary of cluster
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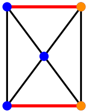

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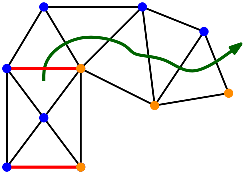

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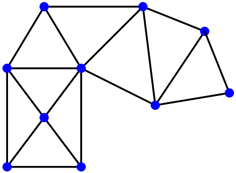

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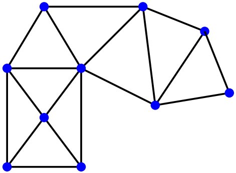

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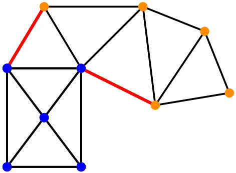

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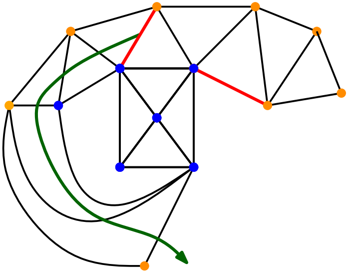

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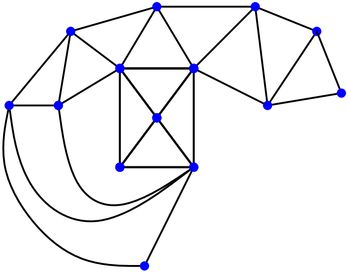

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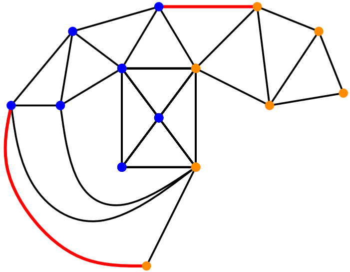

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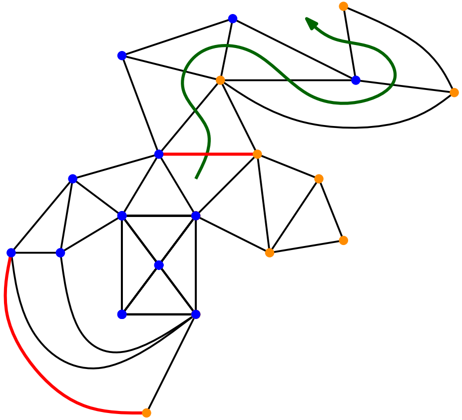

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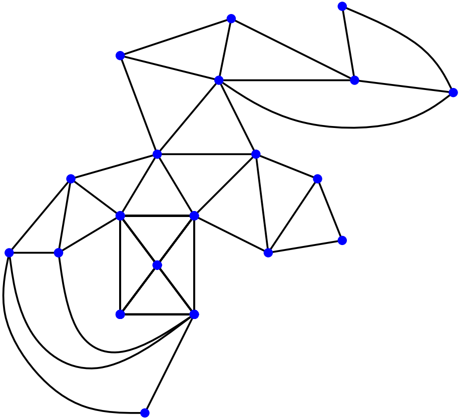

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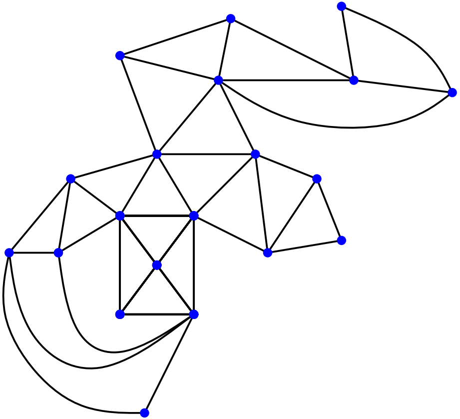

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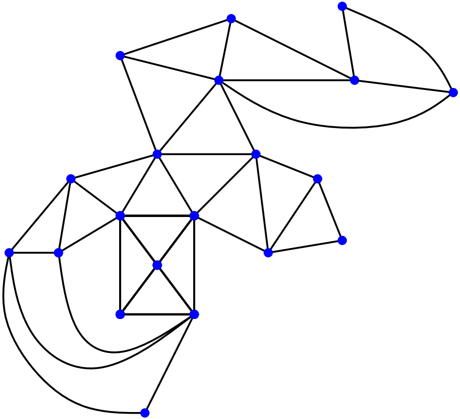

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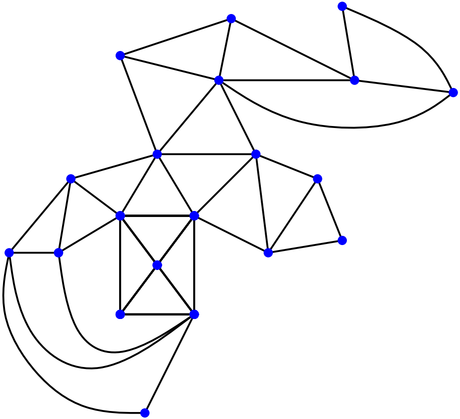
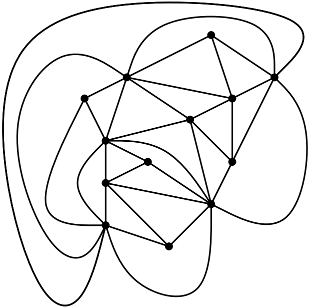


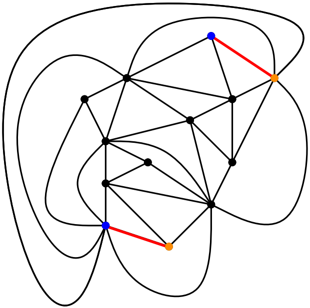
- *This exploration also respects the Markovian structure of the map.*
- If we work on an "infinite" planar map, the conditional law of the map in the unbounded component only depends on the boundary length.
- Expect that at large scales this growth process looks the same as FPP, hence the same as the graph metric ball

Continuum limit ansatz



- Sample a random planar map

Continuum limit ansatz



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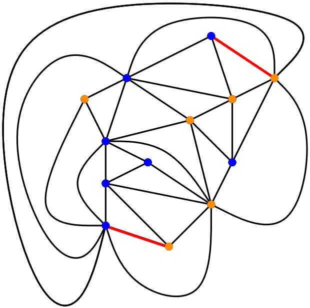

- Sample a random planar map and two edges uniformly at random
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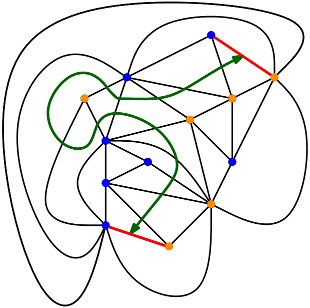

- Sample a random planar map and two edges uniformly at random
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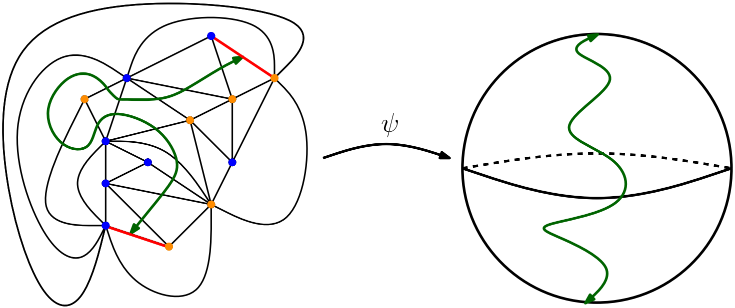
Continuum limit ansatz



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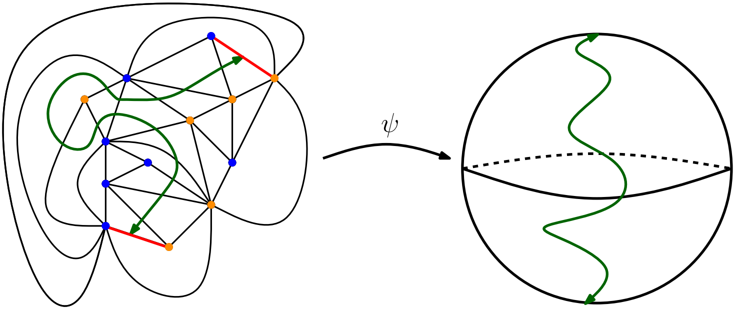

- Sample a random planar map and two edges uniformly at random
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Ansatz Image of random map converges to a $\sqrt{8/3}$-LQG surface and the image of the interface converges to an independent SLE_6.

Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point x

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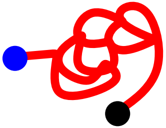

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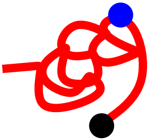

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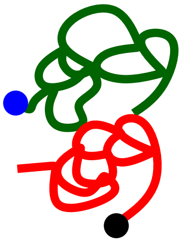

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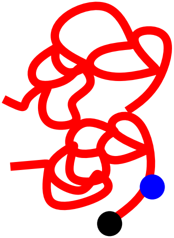

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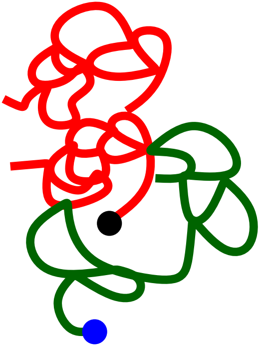

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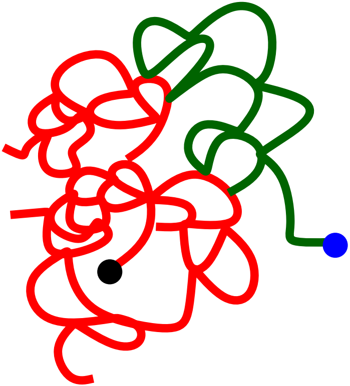

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$\mathrm{QLE}(8/3, 0)$ is the limit as $\delta \to 0$ of this growth process. It is described in terms of a radial Loewner evolution which is driven by a measure valued diffusion.

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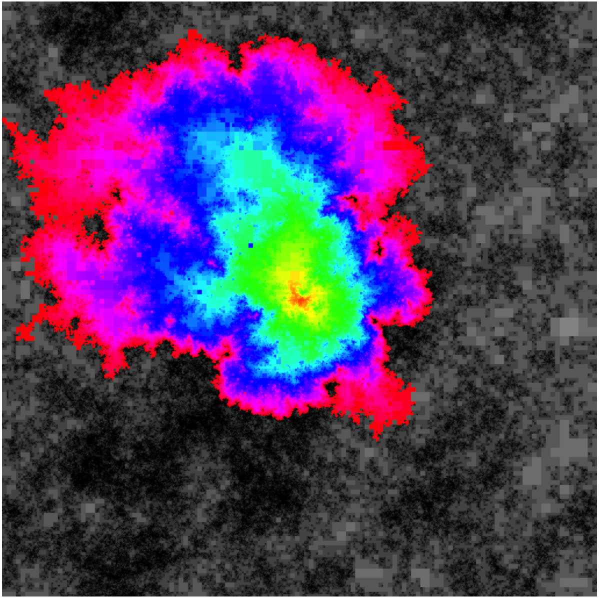
$\mathrm{QLE}(8/3, 0)$ is SLE_6 with **tip re-randomization**.



Discrete approximation of $\mathrm{QLE}(8/3, 0)$. Metric ball on a $\sqrt{8/3}$-LQG

What is $\mathrm{QLE}(\gamma^2, \eta)$?

$\mathrm{QLE}(8/3, 0)$ is a member of a two-parameter family of processes called $\mathrm{QLE}(\gamma^2, \eta)$

- γ is the type of LQG surface on which the process grows
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$$\left(\frac{d\nu}{d\mu}\right)^{\eta} d\mu$$

where ν (resp. μ) represents harmonic (resp. length) measure.

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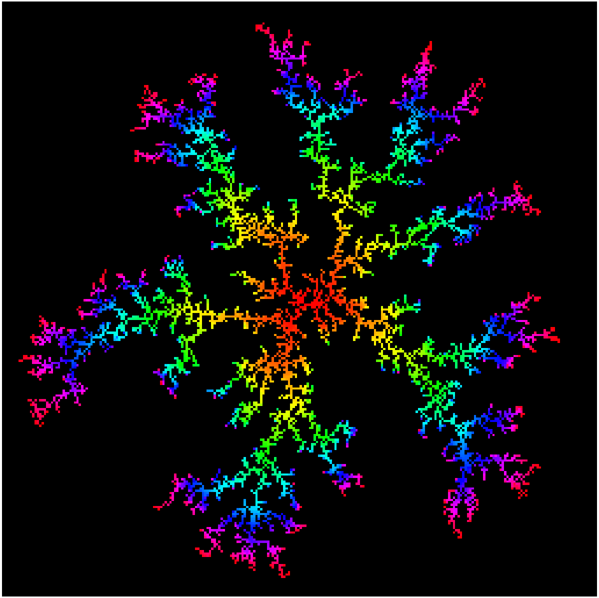
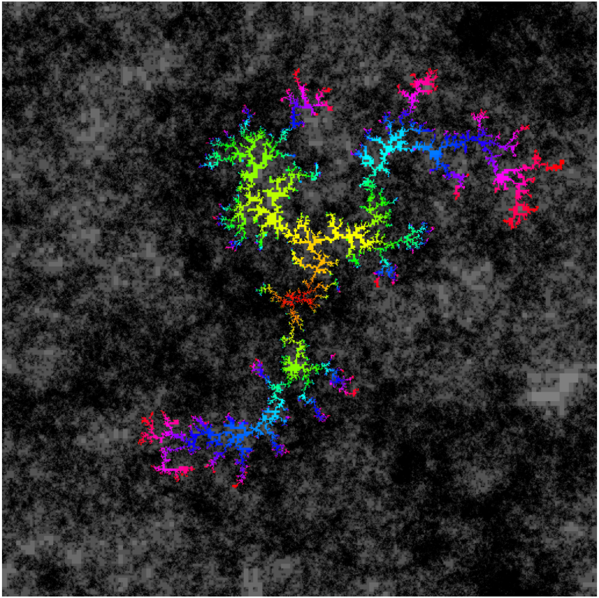
- **First passage percolation:** $\eta = 0$
- **Diffusion limited aggregation:** $\eta = 1$
- **η-dieletric breakdown model:** general values of η



Euclidean DLA



Discrete approximation of $\mathrm{QLE}(2,1)$. DLA on a $\sqrt{2}$-LQG

Diffusion limited aggregation

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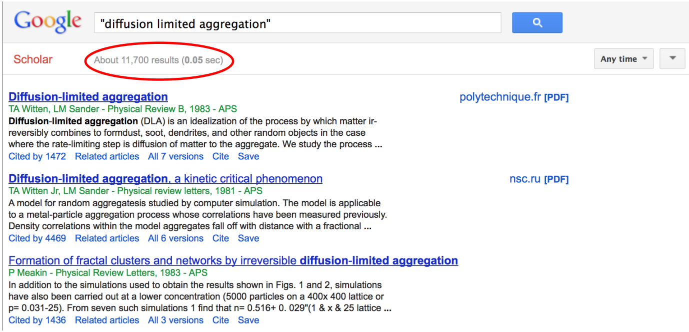

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TA Witten, LM Sander - Physical Review B, 1983 - APS
Diffusion-limited aggregation (DLA) is an idealization of the process by which matter ir-reversibly combines to formdust, soot, dendrites, and other random objects in the case where the rate-limiting step is diffusion of matter to the aggregate. We study the process ...
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P Meakin - Physical Review Letters, 1983 - APS
In addition to the simulations used to obtain the results shown in Figs. 1 and 2, simulations have also been carried out at a lower concentration (5000 particles on a 400x 400 lattice or p= 0.031-25). From seven such simulations 1 find that n= 0.516+ 0. 029"(1 & x & 25 lattice ...
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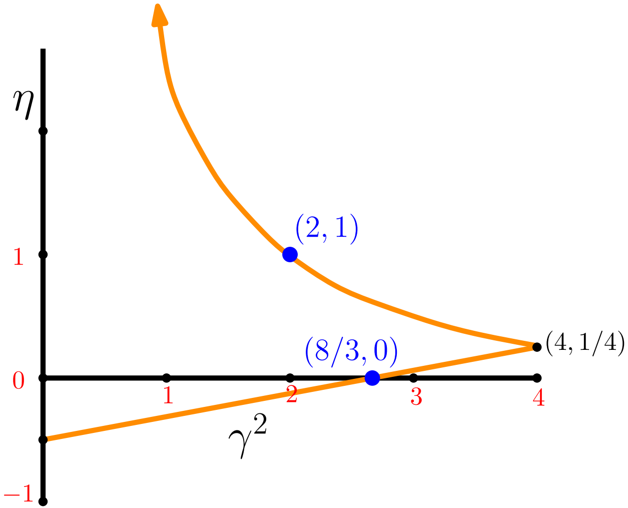
Schramm 2006 ICM proceedings:

Given that the fractals produced by DLA are not conformally invariant, it is not too surprising that it is hard to faithfully model DLA using conformal maps. Harry Kesten [44] proved that the diameter of the planar DLA cluster after n steps grows asymptotically no faster than $n^{2/3}$, and this appears to be essentially the only theorem concerning two-dimensional DLA, though several very simplified variants of DLA have been successfully analysed.

$\mathrm{QLE}(\gamma^2, \eta)$ processes we can construct



Each of the $\mathrm{QLE}(\gamma^2, \eta)$ processes with (γ^2, η) on the orange curves is built from an SLE_κ process using tip re-randomization.

Results

What we can do:

- Existence of $\mathrm{QLE}(\gamma^2, \eta)$ on the orange curves as a Markovian exploration of a γ-LQG surface.
- Derive an SPDE which the measure valued diffusion satisfies
- Continuity of the outer boundary of the growth at a given time
- Phases for sample path behavior: which QLEs are trees, have holes, and fill space

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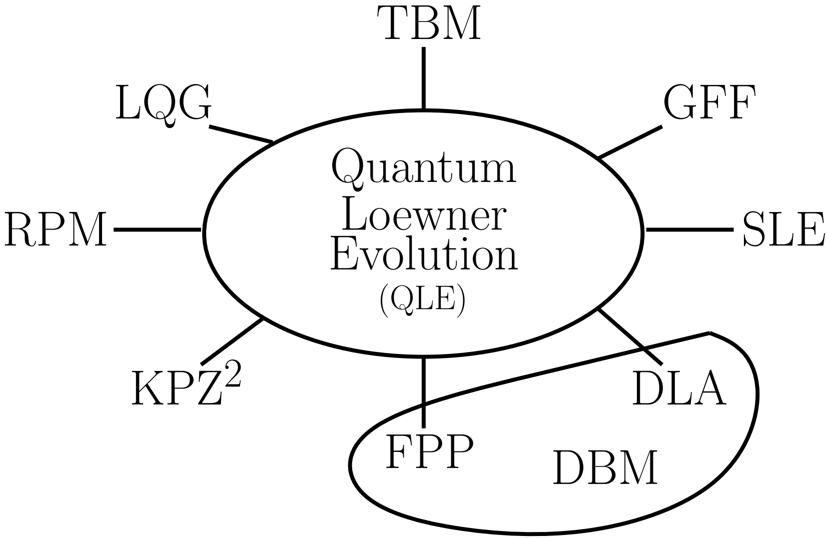
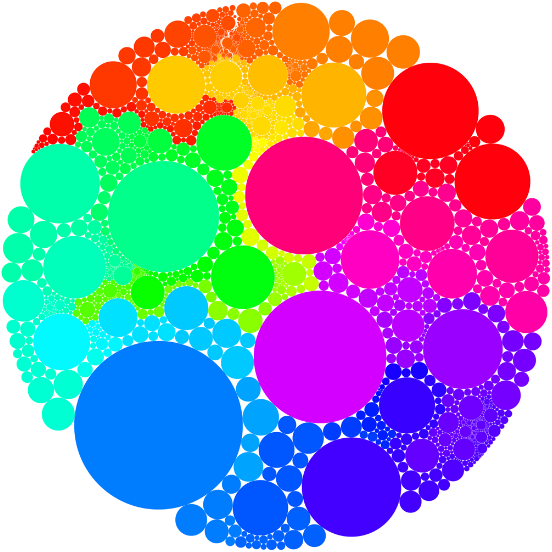
What we would like to do: construct and study $\mathrm{QLE}(\gamma^2, \eta)$ for (γ^2, η) pairs off the orange curves

QLE is connected to other topics in probability





Thanks!