Convergence of percolation on random quadrangulations

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May 22, 2017

Outline

Part I: Introduction — percolation and random planar maps

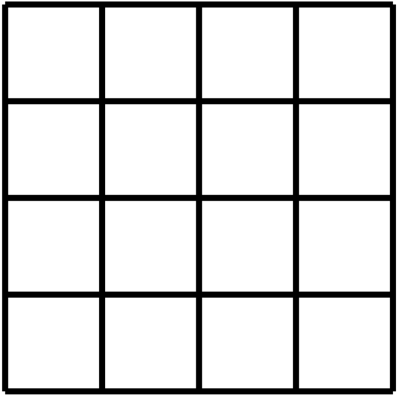
Part II: SLE_6 **on Brownian surfaces**

Part III: Proof ideas

Part I: Introduction

Percolation review

- Graph $G = (V, E)$, $p \in (0, 1)$.



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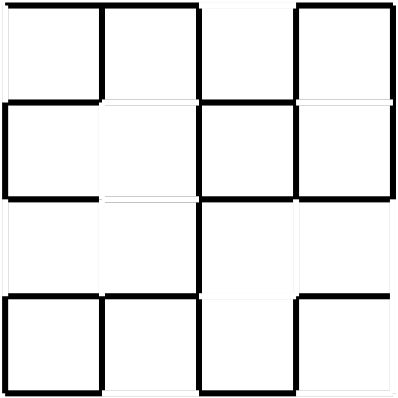

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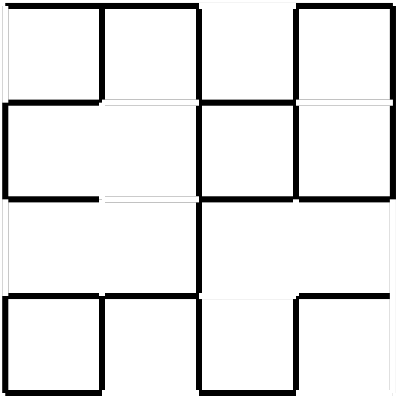

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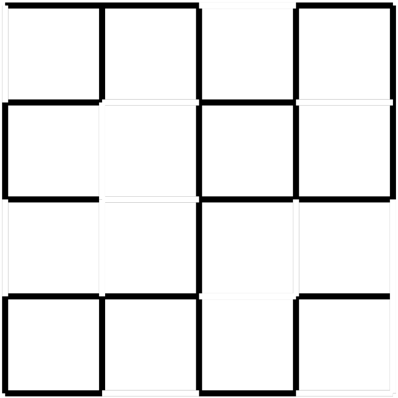

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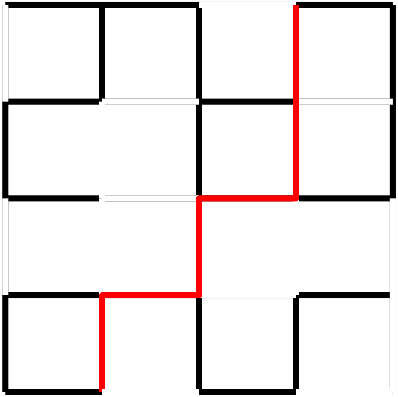

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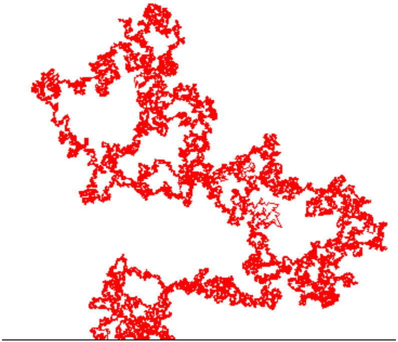

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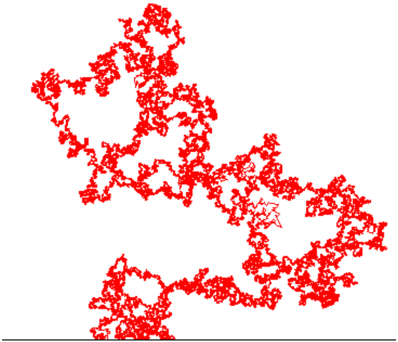
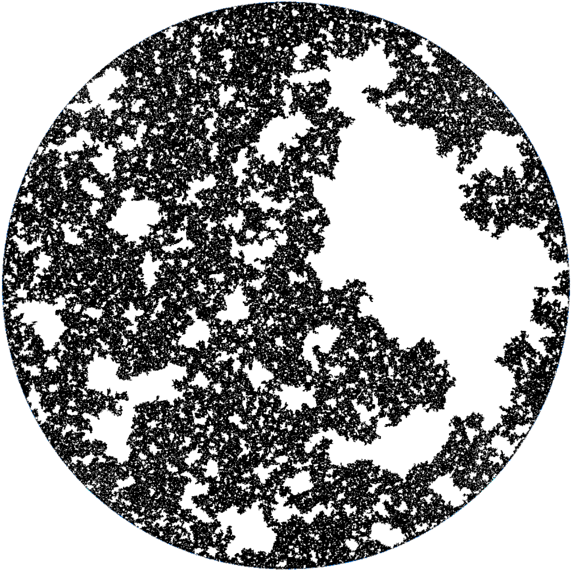


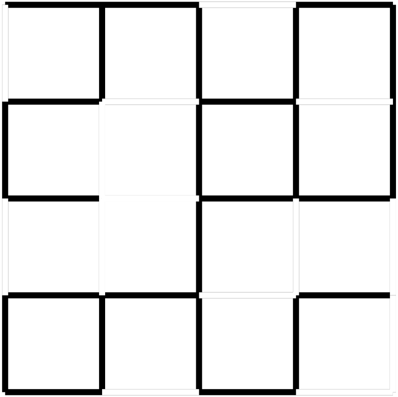
Variants: site percolation, face percolation, etc...



Critical bond percolation on a box in $\mathbf{Z}^2$ with side-length 1000, conformally mapped to $\mathbf{D}$. Shown are the clusters which touch the boundary.

Results on planar lattices

- **Kesten:** $p_c = \frac{1}{2}$ for bond percolation on the □-lattice



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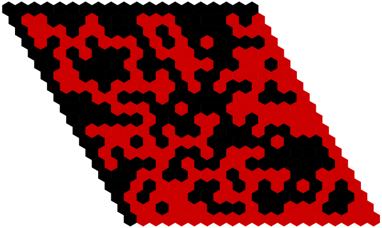

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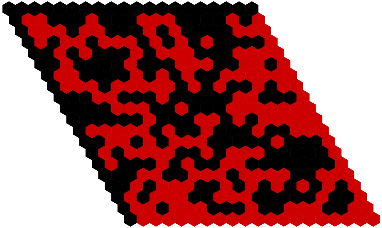

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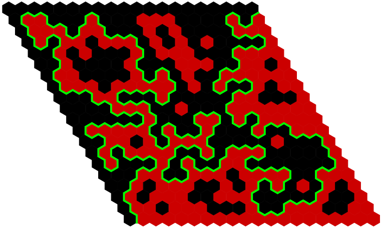

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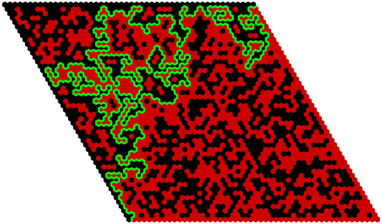

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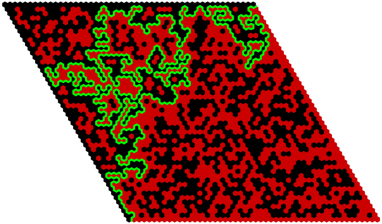

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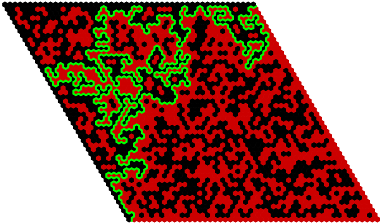

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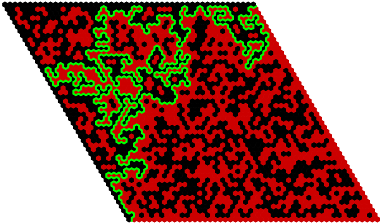

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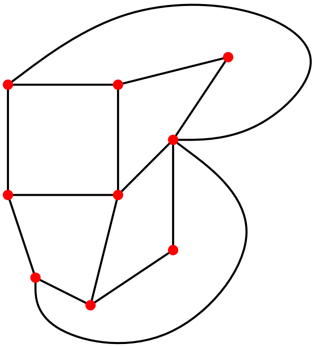
This talk is about proving the convergence of percolation on *random planar maps*.

Random planar maps



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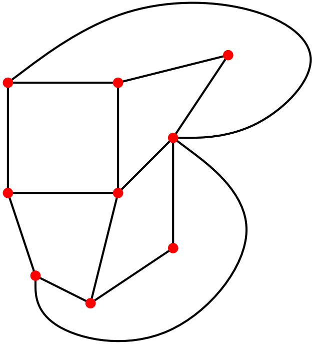

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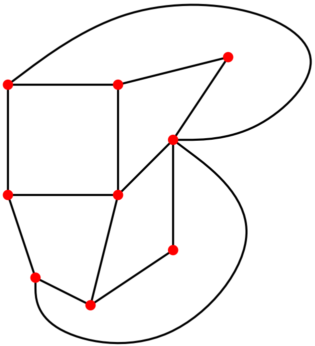

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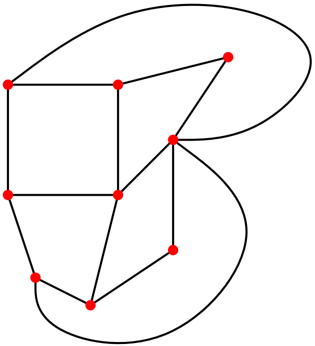

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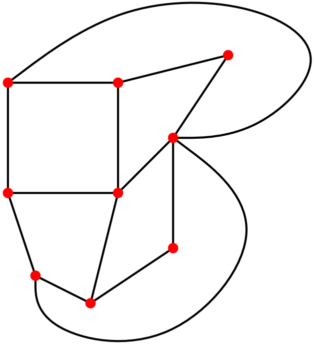

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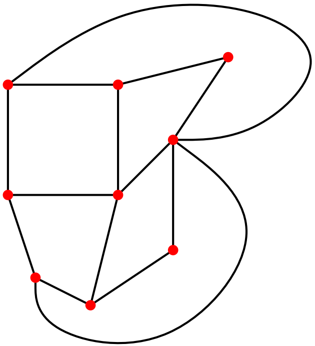

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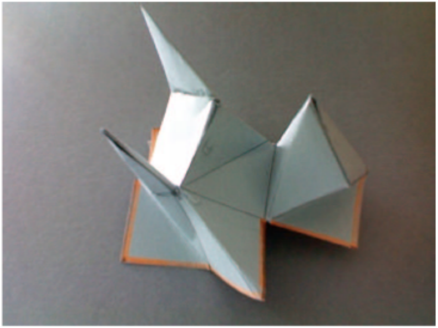
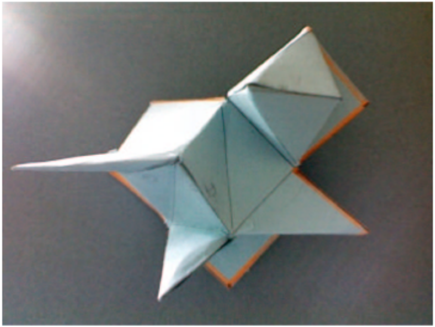
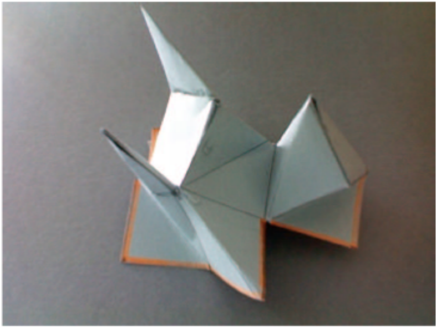
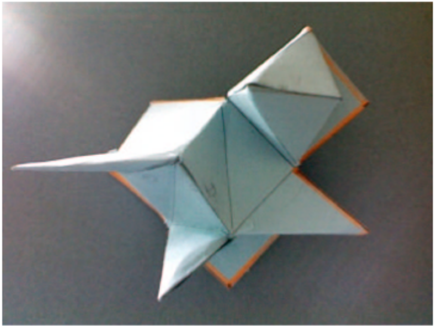
- Interested in **uniformly random quadrangulations** with n faces — **random planar map** (RPM).
- First studied by Tutte in 1960s while working on the four color theorem
  - **Combinatorics**: enumeration formulas
  - **Physics**: statistical physics models: percolation, Ising, UST ...
  - **Probability**: "uniformly random surface," Brownian surface

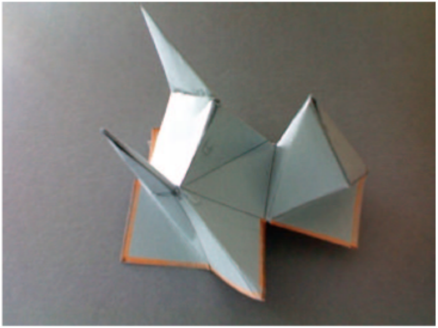
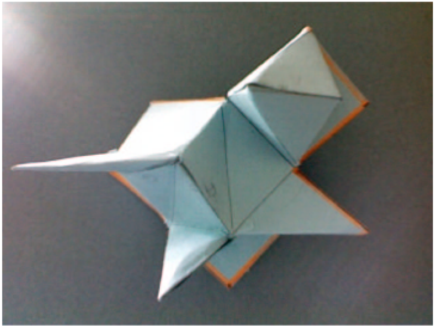








What is the structure of a typical quadrangulation when the number of faces is large?

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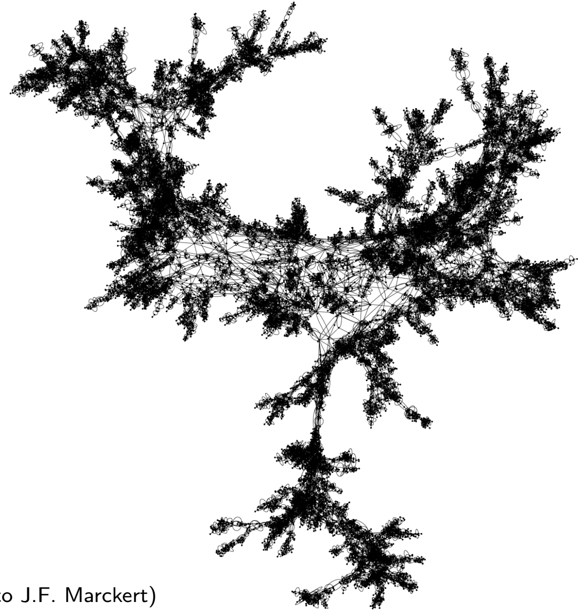
How many are there? **Tutte:**

$$\frac{2 \times 3^n}{(n+1)(n+2)} \binom{2n}{n}.$$

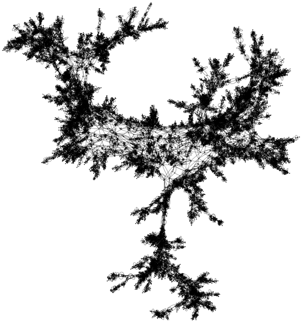
Random quadrangulation with 25,000 faces



(Simulation due to J.F. Marckert)

Topologies for quadrangulations

It is natural to consider □'s with different topologies



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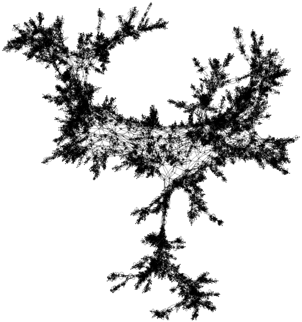

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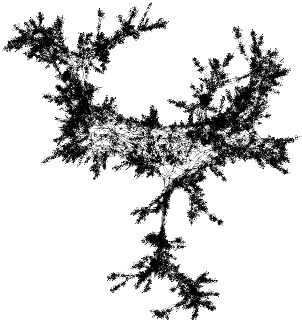

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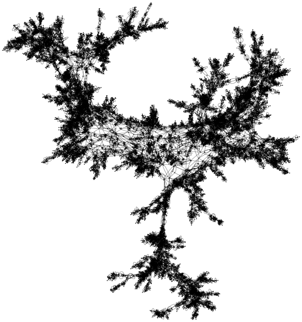

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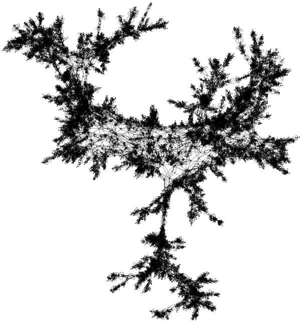

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- **Gromov-Hausdorff-Prokhorov-uniform**: metric space + measure + path

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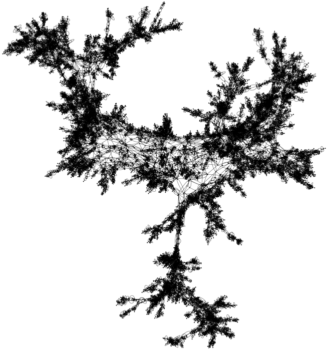
Large scale structure of random quadrangulations



(Simulation due to J.F. Marckert)

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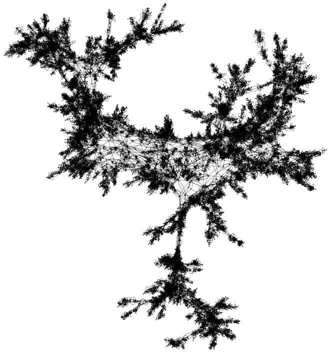

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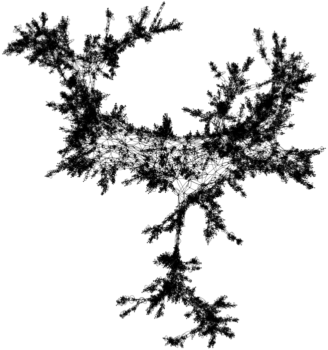

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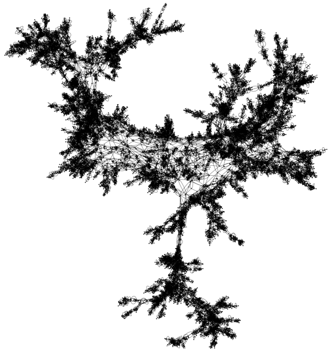

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- Subsequentially limiting space is a.s.:
  - 4-dimensional (Le Gall)
  - homeomorphic to the 2-sphere (Le Gall and Paulin, Miermont)

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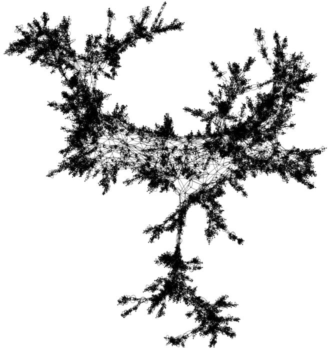

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- Rescaling by $n^{-1/4}$ gives a tight sequence of metric spaces (Le Gall)
- Subsequentially limiting space is a.s.:
  - 4-dimensional (Le Gall)
  - homeomorphic to the 2-sphere (Le Gall and Paulin, Miermont)
- There exists a unique limit in distribution: **the Brownian map** (Le Gall, Miermont)

Convergence results toward Brownian surfaces

General principle: Uniformly random planar □'s with n faces with distances rescaled by $n^{-1/4}$ converge to Brownian surfaces in the Gromov-Hausdorff-Prokhorov topology (metric space + measure).

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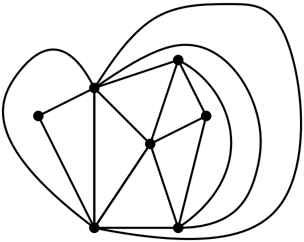
- □ of the disk (simple boundary, random area) $\to$ Brownian disk (Gwynne-M.)

Percolation on random planar maps

- **Angel:** $p_c = \frac{1}{2}$ for site percolation on a random $\triangle$



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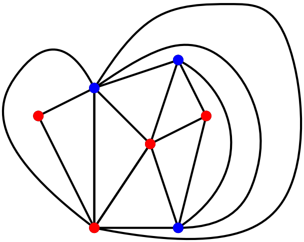

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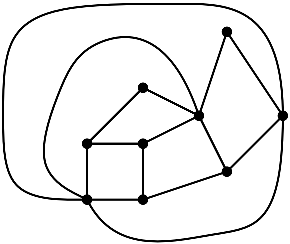

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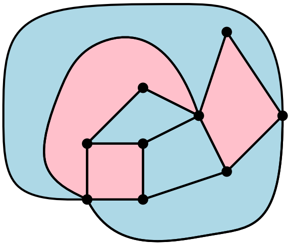

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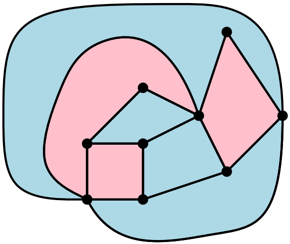

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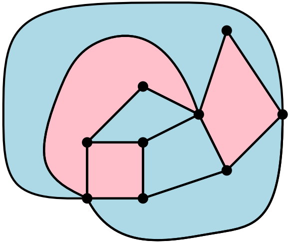

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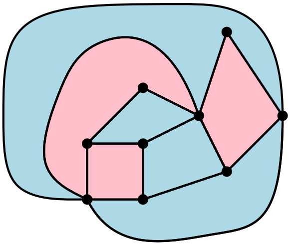

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We will consider critical $p = p_c = \frac{3}{4}$ face percolation on a random $\square$.

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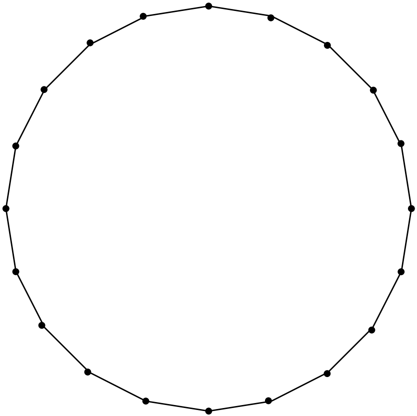

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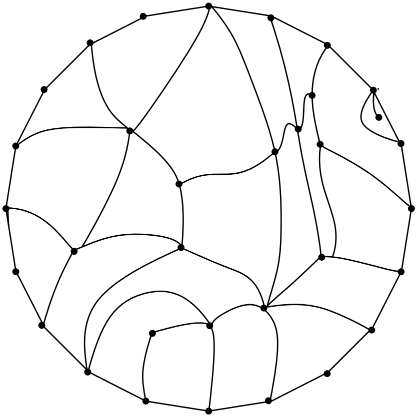

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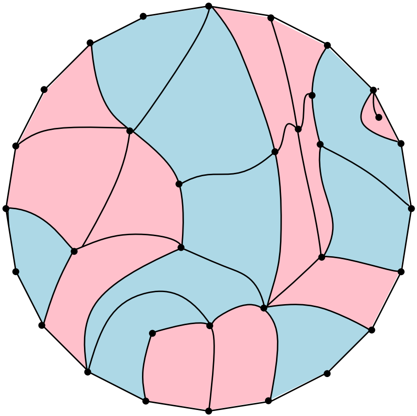

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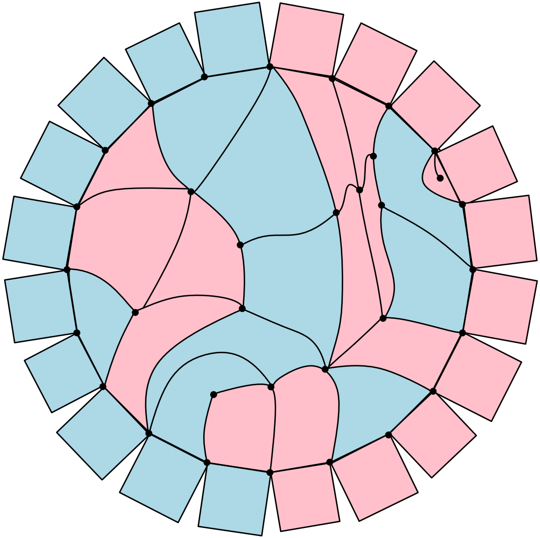

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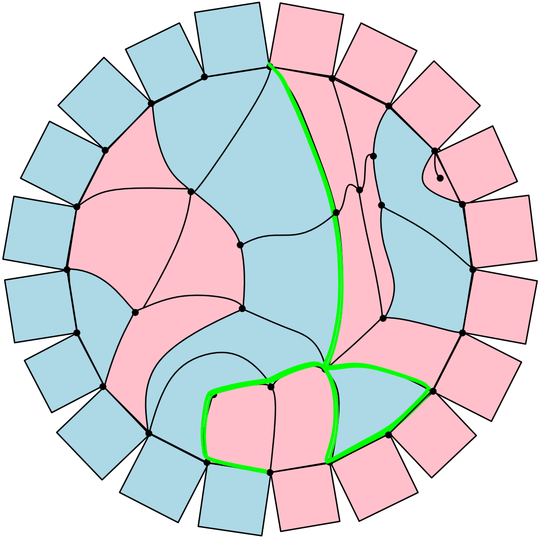

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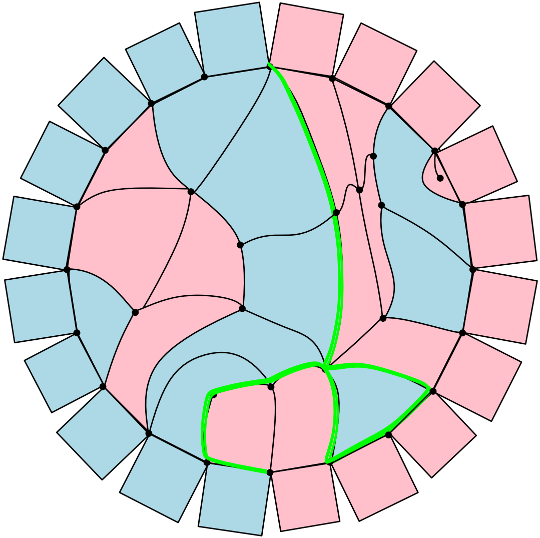
- **Perspective:** It is a *random path* on a *random metric space*

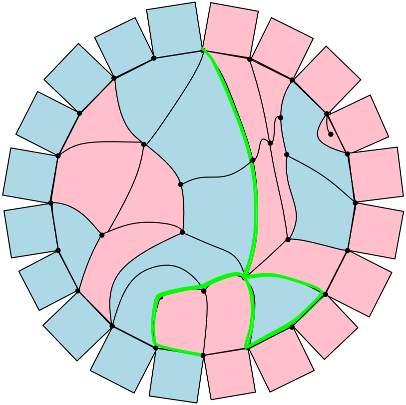


Main result

Theorem (Gwynne-M.)

The exploration path for critical face percolation on a random $\square$ *of the disk with boundary length* 2ℓ *converges as* $\ell \to \infty$ *to a random path on a random metric space with respect to the Gromov-Hausdorff-Prokhorov-uniform topology.*

The limit is SLE_6 *on a Brownian disk.*



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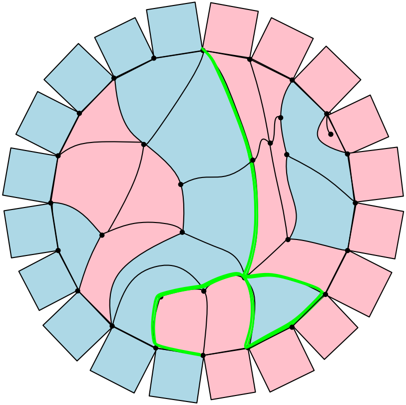

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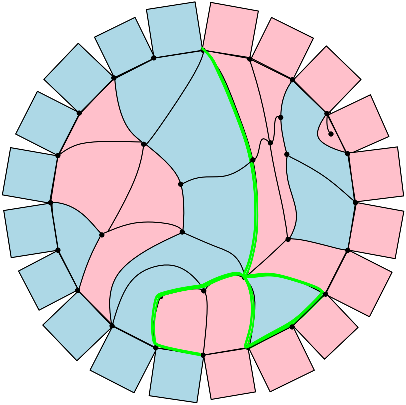

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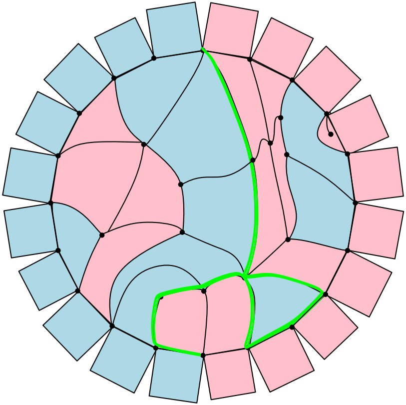


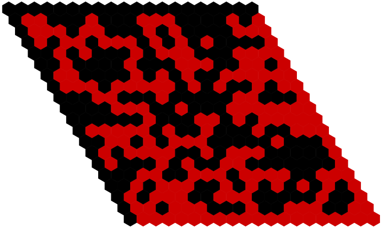
Comments:

- Universal strategy: works for any random planar map model provided one has certain technical inputs.
- Works for other topologies (sphere, plane, half-plane).

Part II: SLE_6 on a Brownian surface

Schramm-Loewner evolution (SLE)

- Random fractal curve in a planar domain



Critical percolation, hexagonal lattice

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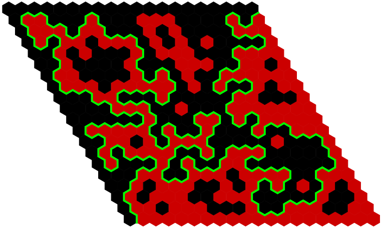

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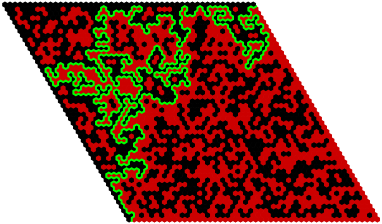

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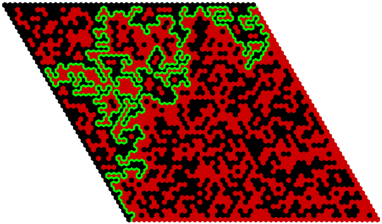

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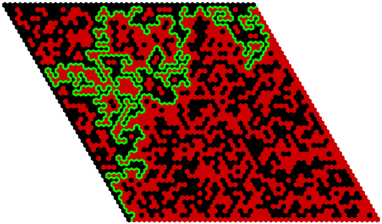

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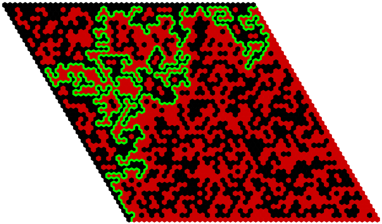

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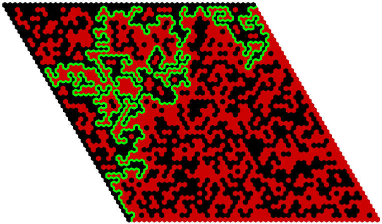

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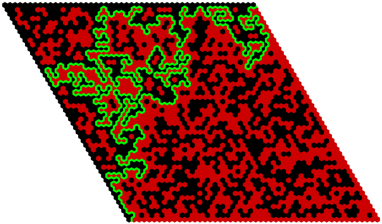
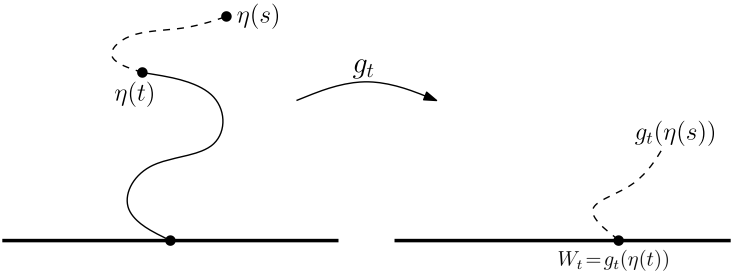
- Some special κ values:
  - $\kappa = 2$ LERW, $\kappa = 8$ UST
  - $\kappa = 8/3$ SAW
  - $\kappa = 3$ Ising, $\kappa = 16/3$ FK-Ising
  - $\kappa = 4$ GFF level lines
  - $\kappa = 6$ Percolation
  - $\kappa = 12$ Bipolar orientations
  - $\cdots$



Critical percolation, hexagonal lattice

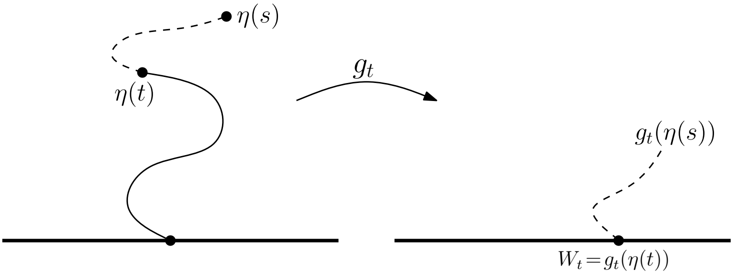
SLE$_\kappa$



Loewner's equation: if η is a non self-crossing path in $\mathbf{H}$ with $\eta(0) \in \mathbf{R}$ and g_t is the Riemann map from the unbounded component of $\mathbf{H} \setminus \eta([0,t])$ to $\mathbf{H}$ normalized by $g_t(z) = z + o(1)$ as $z \to \infty$, then

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t} \text{ where } g_0(z) = z \text{ and } W_t = g_t(\eta(t)). \qquad (\bigstar)$$

SLE_κ



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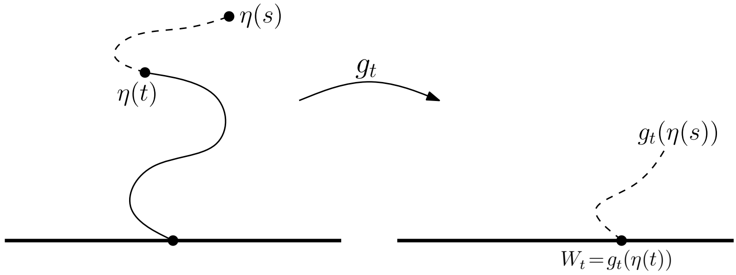

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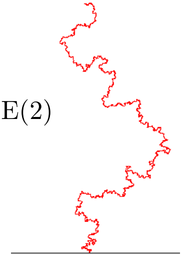
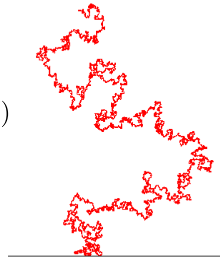
SLE_κ **in H**: The random curve associated with ($\bigstar$) with $W_t = \sqrt{\kappa} B_t$, B a standard Brownian motion. Other domains: apply conformal mapping.

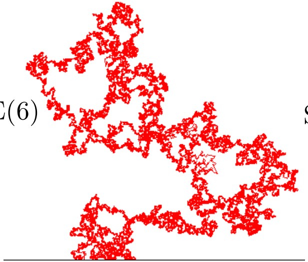
SLE(2)



SLE(4)



SLE(6)



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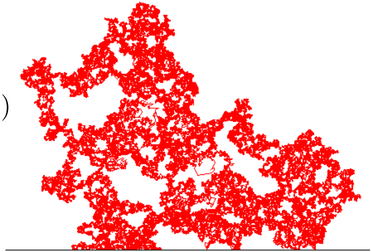

Simulations due to Tom Kennedy.

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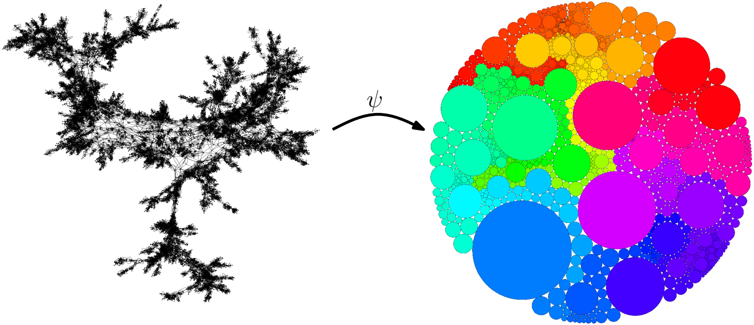
- This is necessary to define SLE_6 on a Brownian surface

Embedding Brownian surfaces into **C**

- It is conjectured that if one takes a uniformly random planar map and then embeds it "conformally" into **C** (using, e.g., circle packing) then the maps will converge to an embedding of the limiting Brownian surface into **C**.



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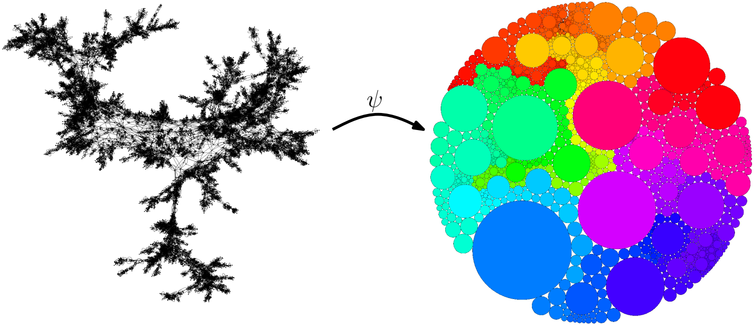

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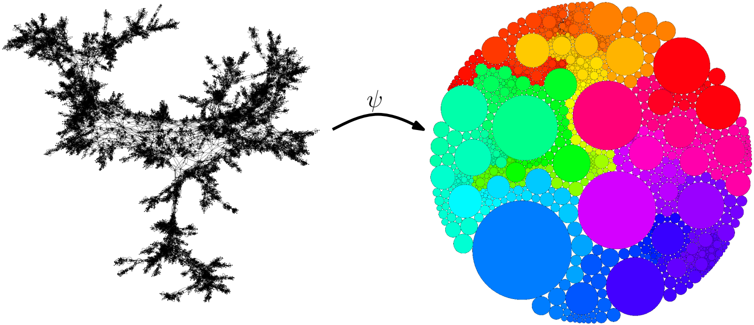

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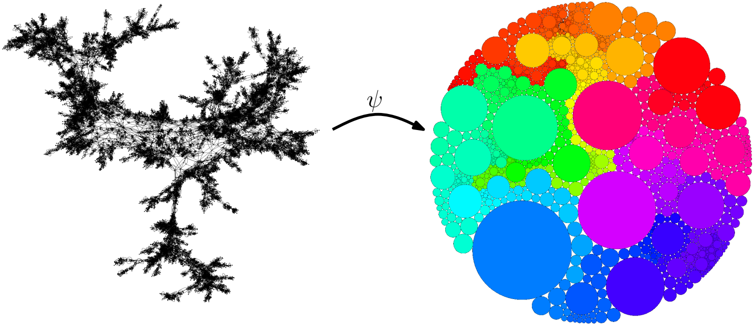

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- Define SLE_6 on a Brownian surface using the $\mathrm{QLE}(8/3, 0)$ embedding.
- Is this the right definition? It is if it is the scaling limit of percolation ...

Part III: Proof ideas

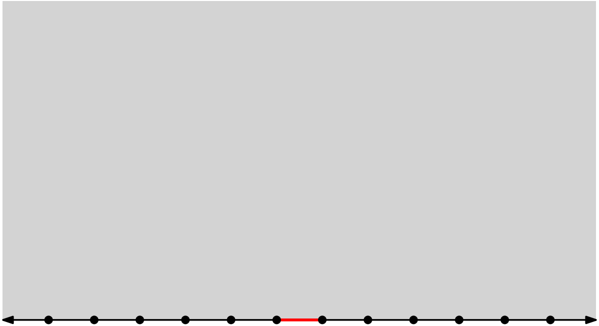
Proof overview

Proof has two steps:

- Construct subsequential limits of the percolation exploration
- Characterization theorem which singles out SLE_6 on a Brownian surface

Peeling exploration

A "peeling" of a map is a Markovian "exploration" in which faces are revealed one at a time.



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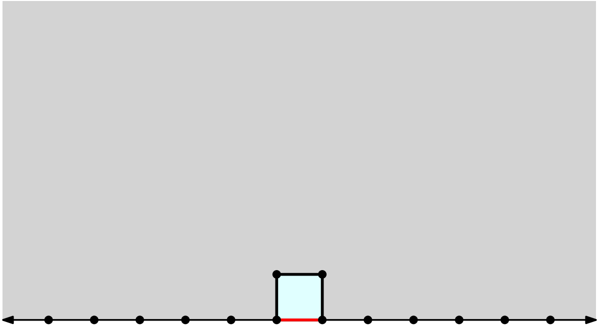

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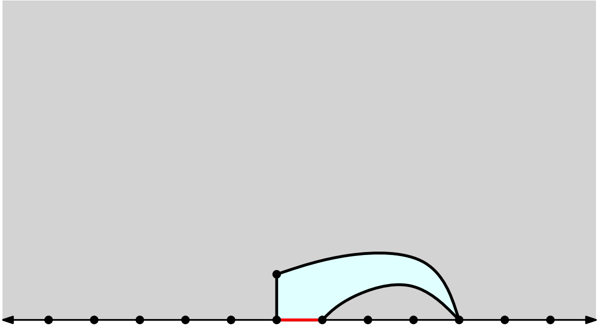

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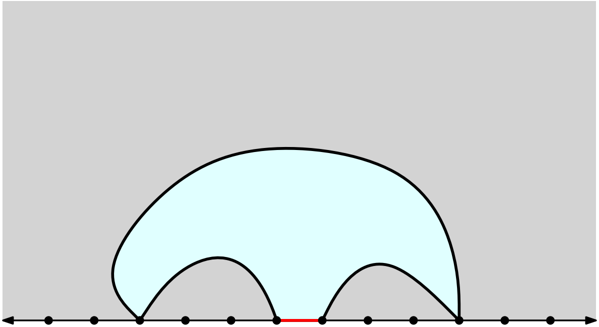

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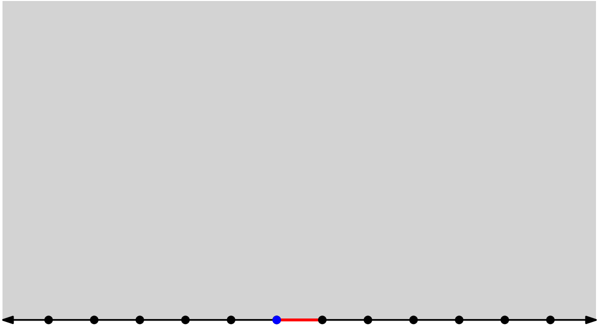

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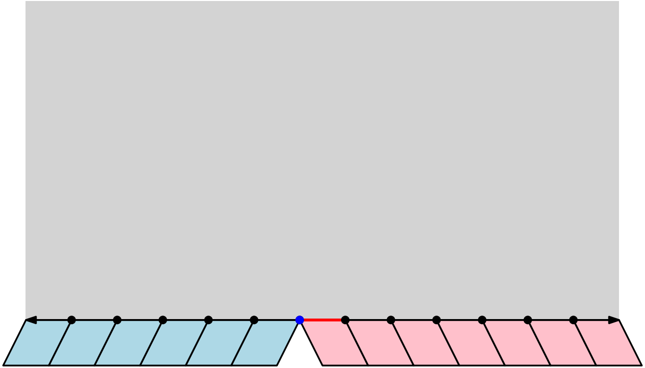

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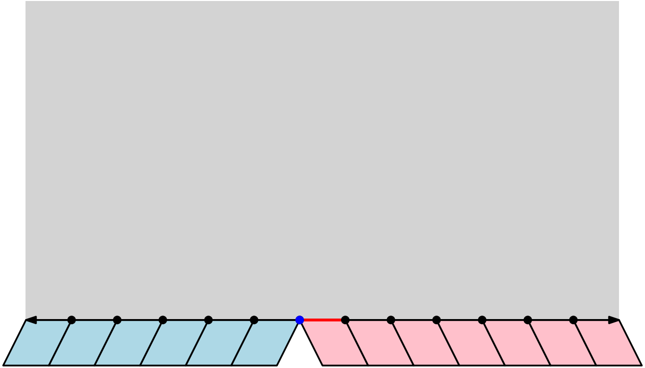

Always know the law of the unexplored region. Impose open/closed ∂-conditions. Choose $\square$'s to reveal so as to explore the percolation interface.

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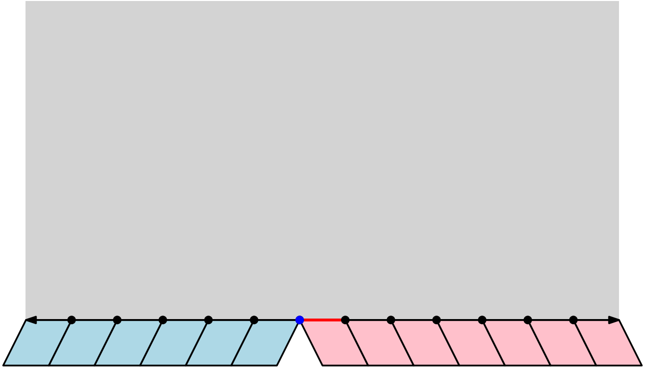

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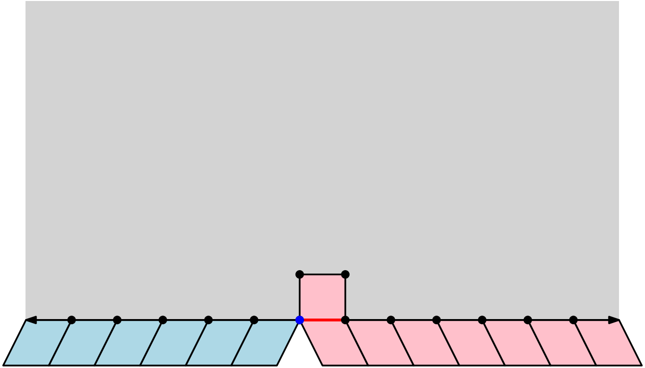

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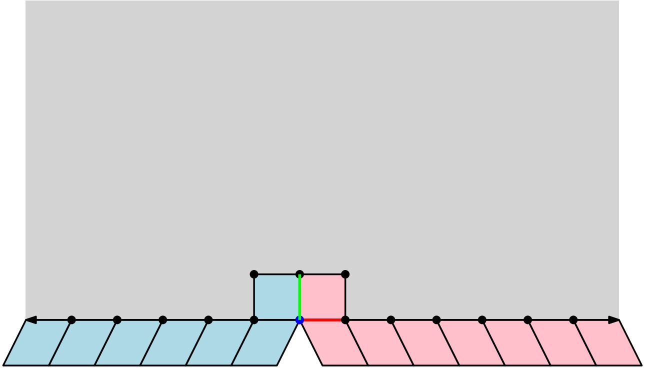

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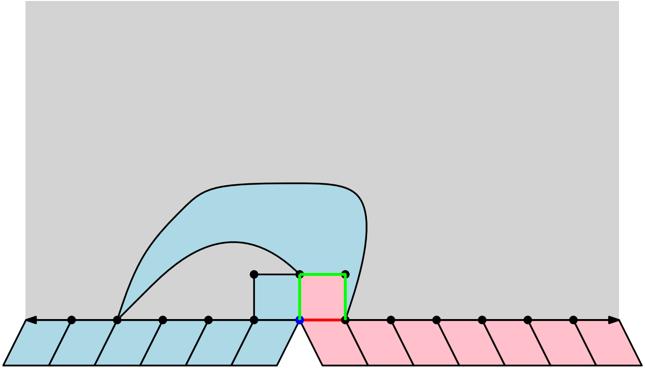

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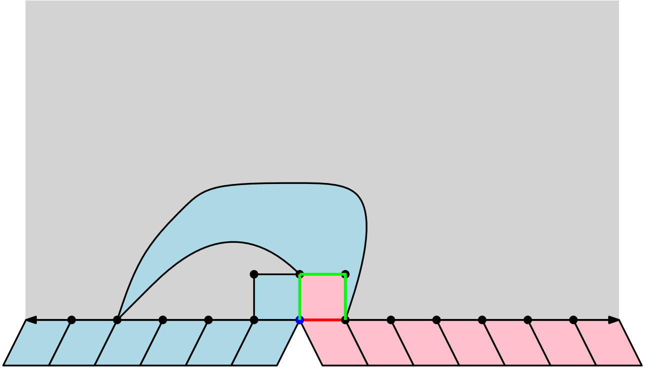

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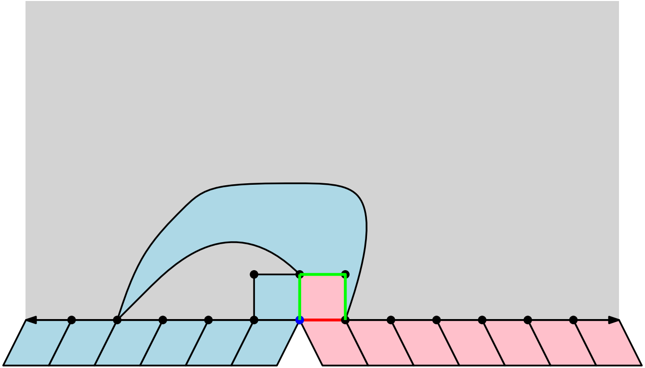

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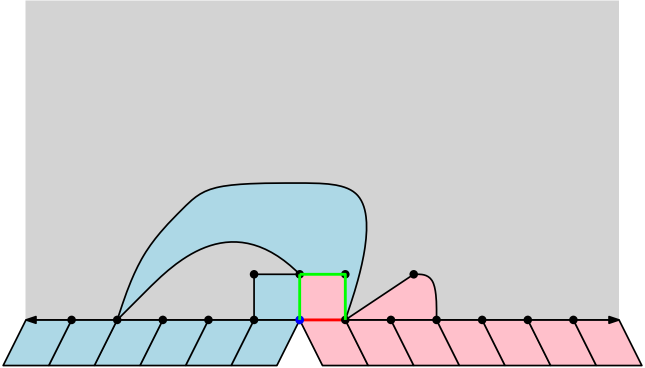

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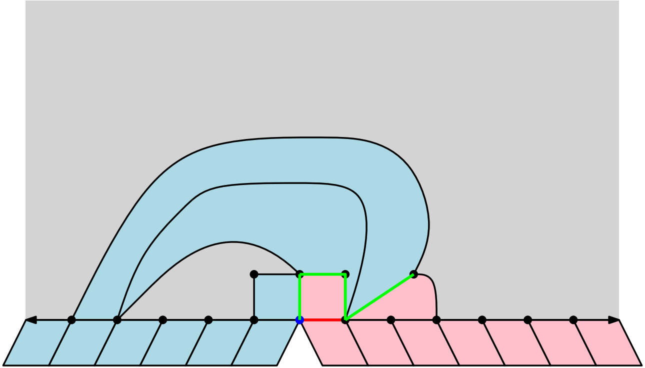

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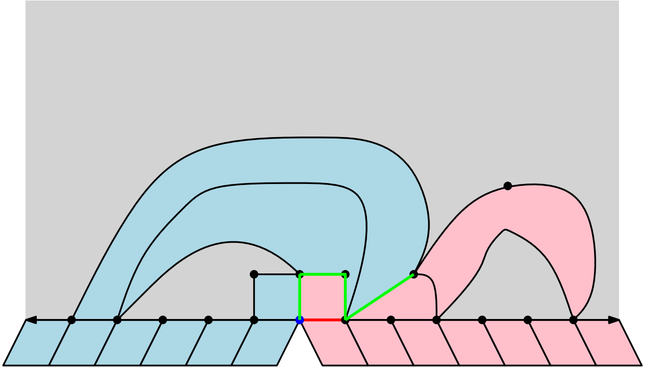

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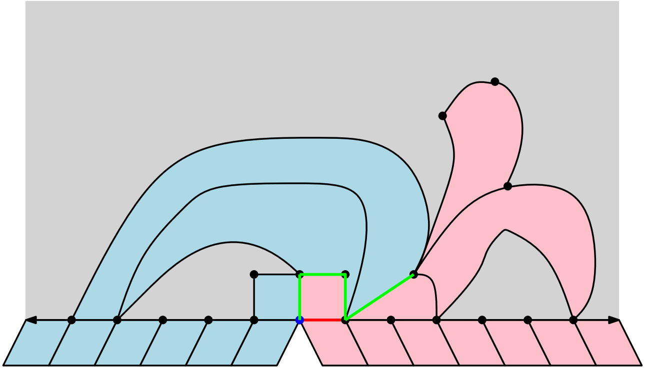

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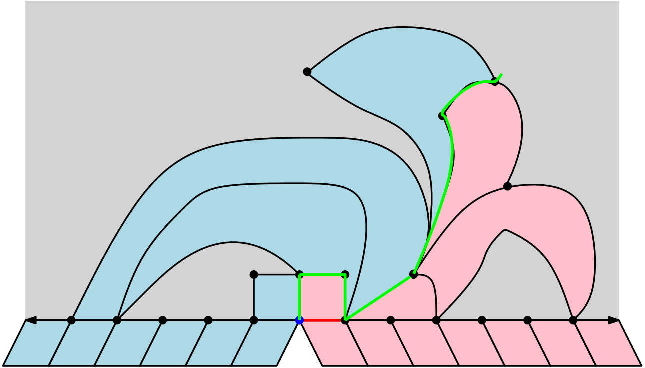

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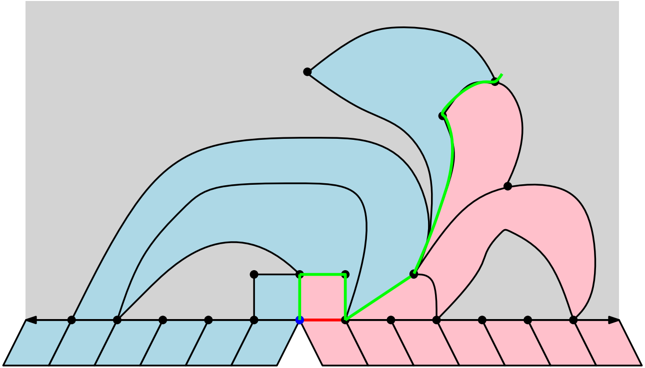

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- It turns out that these three properties characterize SLE_6 on a Brownian surface
  - Proved using the connection between Brownian surfaces and Liouville quantum gravity / GFF

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Convergence results for planar maps (RPM) decorated with a statistical physics model to SLE on a random surface.

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Peanosphere sense (Duplantier, M., Sheffield)

- FK-weighted RPM with $q \in (0,4)$
  - Infinite volume (Sheffield)
  - finite volume (Gwynne, Mao, Sun and Gwynne, Sun)
- Bipolar orientation decorated RPM (Kenyon, M., Sheffield, Wilson)
- Active spanning tree decorated RPM (Gwynne, Kassel, M., Wilson)
- Schnyder woods (Li, Sun, Watson)

Thanks!