ESTIMATES ON THE FUNDAMENTAL SOLUTION TO HEAT FLOWS WITH UNIFORMLY ELLIPTIC COEFFICIENTS

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ABSTRACT

We obtain estimates, from above and below, on the fundamental solution to the Cauchy initial value problem for parabolic equations having uniformly elliptic coefficients. We impose minimal regularity requirements on the coefficients. The estimates are expressed in terms of an energy functional associated with the coefficients.

1. *Introduction*

During the past few years, several authors [**2, 6, 3**], have applied semigroup techniques to re-prove and refine the heat kernel estimates obtained originally by D. Aronson [**1**]. In particular, E. B. Davies introduced a technique which enabled him to sharpen the upper bound in Aronson's estimates by replacing Aronson's qualitative bound, in terms of the Euclidean distance, with a bound which is expressed in terms of the Riemannian distance associated with second-order coefficients under consideration. Shortly after Davies, E. Fabes and D. Stroock [**3**] discovered that the ideas of J. Nash in [**4**] can be used to not only get Davies' upper bound but also Aronson's lower bound. Unfortunately, the lower bound in [**3**] is, like Aronson's, qualitative and therefore is not an entirely satisfactory complement to Davies' upper bound. Furthermore, in both [**2**] and [**3**], the operators governing the heat flow have to be in divergence form (in [**2**] the coefficients are independent of time, as well). A plan for removing this restriction is outlined in § 2 of [**5**], but nothing is done there to improve the lower bound. The purpose of the present article is to carry out the programme outlined in [**5**] and, at the same time, to sharpen the lower bound.

We will now introduce the notation which we will use throughout. Let L be a time-dependent second-order differential operator on functions on $\mathbb{R}^N$. For maximal symmetry in taking L^2-adjoint we write L in the form

$$L = \nabla \cdot (a(t, x)\nabla) + ab(t, x) \cdot \nabla - \nabla \cdot (a\hat{b}(t, x)) + c(t, x). \tag{1.1}$$

The coefficients a, b, $\hat{b}$ and c of L are measurable functions on $\mathbb{R} \times \mathbb{R}^N$, a taking its values in the set of $N \times N$ positive-definite symmetric matrices, b and $\hat{b}$ in $\mathbb{R}^N$, and c in $\mathbb{R}$. Our concern is with the *propagator* $P_{t,u}$, for $t < u$, associated with L and with its transition density $p(t, x; u, y)$, the *heat kernel* for L. Thus, at least formally,

$$\left(\frac{\partial}{\partial t} + L\right) P_{t,u} = 0,$$

(1.2)

$$\lim_{t \nearrow u} P_{t,u} = \text{Identity},$$

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and

$$(1.3)\qquad (P_{t,u}\phi)(x) = \int_{\mathbb{R}^N} p(t, x\,; u, y)\phi(y)\,dy.$$

When certain coefficients fail to be differentiable, L has to be interpreted in a weak sense (cf. [**5**, § 3] for more details): for test-functions ϕ and ψ on $\mathbb{R}^N$ and some fixed $u \in \mathbb{R}$, the meaning of (1.2) is that the function $t \in (-\infty, u] \mapsto \phi_t \equiv P_{t,u}\phi$ satisfies

$$(1.4)\qquad \left(\psi, \frac{\partial \phi}{\partial t}\right) = (\nabla\psi, \nabla\phi_t)_a - (\psi b, \nabla\phi_t)_a - (\nabla\psi, \hat{b}\phi_t)_a - (\psi, c\phi_t).$$

Here $(\ ,\ )$ denotes the L^2 inner product in $\mathbb{R}^N$, and $(\ ,\ )_a$ the L^2 inner product for vector-valued functions, using a for the inner product on $\mathbb{R}^N$. Thus

$$(\nabla\psi, \nabla\phi_t)_a = \int_{\mathbb{R}^N} \langle \nabla\psi, \nabla\phi_t\rangle_a\,dx = \int_{\mathbb{R}^N} \langle \nabla\psi(x), a(t, x)\,\nabla\phi(t, x)\rangle\,dx.$$

Equation (1.4) fully characterises the relation between the propagator and its coefficients and is the starting point for all our analysis. Our aim is to find upper and lower bounds on $p(t, x\,; u, y)$ relying only on the uniform positivity of a and on bounds on each of the coefficients: for simplicity we suppose there are constants $\lambda \in [1, \infty)$ and $\Lambda \in [0, \infty)$ such that, uniformly on $\mathbb{R} \times \mathbb{R}^N$,

$$(1.5)\qquad \lambda^{-1}I \leq a \leq \lambda I \quad\text{and}\quad |b|_a^2 + |\hat{b}|_a^2 + |c| \leq \Lambda,$$

where the bounds on a hold in the sense of symmetric matrices, and where $|b|_a^2 = \langle b, ab\rangle$.

The upper bounds we obtain are expressed in terms of an energy function: for $t, u \in \mathbb{R}$ with $t < u$ and $x, y \in \mathbb{R}^N$, define

$$\Gamma(t, x\,; u, y) = \left\{\gamma \in C([t, u], \mathbb{R}^N)\colon\ \gamma_t = x,\ \gamma_u = y \text{ and } \int_t^u |\dot{\gamma}_s|^2\,ds < \infty\right\}$$

and

$$E(t, x\,; u, y) = \inf_{\gamma\in\Gamma(t,x;\,u,y)} \frac{1}{4}\int_t^u |\dot{\gamma}_s - a(b - \hat{b})|_{a^{-1}}^2\,(s, \gamma_s)\,ds.$$

When a is independent of time and $b = \hat{b}$, we have

$$(1.6)\qquad E(t, x\,; u, y) = d(x, y)^2/4(u - t),$$

where d is the distance function on $\mathbb{R}^N$ associated with the Riemannian metric a^{-1}. In the lower bounds it is a modification of E that appears: define, for $\beta \in (0, \infty)$,

$$E_\beta(t, x\,; u, y) = \inf_{\gamma\in\Gamma(t,x;\,u,y)} \frac{1}{4}\int_t^u (\rho_{\beta(u-t)} * |\dot{\gamma}_s - a(b - \hat{b})|_{a^{-1}}^2)(s, \gamma_s)\,ds,$$

where $*$ denotes convolution in $\mathbb{R}^N$ and

$$\rho_\tau(z) = (4\pi\tau)^{-N/2}e^{-|z|^2/4\tau}.$$

The following result summarises Theorems 2.7 and 3.5.

THEOREM 1.1. *Suppose that a and $b - \hat{b}$ are uniformly continuous. Then, for every $\alpha \in (\frac{1}{2}, 1)$ satisfying $\alpha^2/(2\alpha - 1) > \lambda^2$, there is a constant $C(\alpha, \lambda, N) \in (0, \infty)$*

such that, for all $t, u \in \mathbb{R}^N$ with $t<u$ and all $x, y \in \mathbb{R}^N$, we have

$$\begin{aligned}(u-t)^{-N/2}&\exp\{-E_\beta(t, x ; u, y) - 2\Lambda(u-t)\\ &\qquad - C[1+\Lambda(u-t)+E_\beta(t, x ; u, y)]^{1/(4\alpha-1)}\}\\ &\leqslant p(t, x ; u, y)\\ &\leqslant \left(\frac{C[1+\Lambda(u-t)+E(t, x ; u, y)]^{1/(2\alpha-1)}}{u-t}\right)^{N/2}\\ &\qquad \times \exp\left\{-E(t, x ; u, y) + \int_t^u \sup_{z\in\mathbb{R}^N} (\tfrac{1}{4}|b+\hat{b}|_a^2 + c)(s, z)\, ds\right\},\end{aligned}$$

for $\beta = (1+\Lambda(u-t)+E(t, x ; u, y))^{-1/(4\alpha-1)}$.

These estimates are strongest when α is near 1. We will see later that estimates corresponding to $\alpha=1$ hold when the energy function is Lipschitz continuous. There are two cases when we know this is true: first when a and $b-\hat{b}$ are Hölder continuous (see Theorems 2.8 and 3.7 for the resulting estimates); second when $b=\hat{b}$ and a depends on position alone, whence E is Lipschitz by virtue of (1.6). We do not set out the estimates for this second case in general, but here is the result for the simplest and most important special case:

THEOREM 1.2. *Suppose that a is independent of time, that $b=\hat{b}=0$ and $c=0$. Then there is a constant $C(\lambda, N)\in(0,\infty)$ such that, for all $t, u\in\mathbb{R}^N$ with $t<u$ and all $x, y\in\mathbb{R}^N$, we have*

$$\begin{aligned}(u-t)^{-N/2}&\exp\{-E_\beta(t, x ; u, y) - C[1+E_\beta(t, x ; u, y)^{\frac{1}{3}}]\}\\ &\leqslant p(t, x ; u, y) \leqslant \left(\frac{C[1+E(t, x ; u, y)]}{u-t}\right)^{N/2}\exp\{-E(t, x ; u, y)\},\end{aligned}$$

for $\beta=(1+E(t, x ; u, y)^{\frac{1}{3}})^{-1}$.

The lower bound in this result is a special case of Theorem 3.7, though one would want to make use of (1.6) rather than the results of the appendix in giving a direct proof. The upper bound, due to E. B. Davies, does not require a to be uniformly continuous and therefore is not implied by Theorem 2.8. A proof of the upper bound as stated, making use of time independence at an earlier stage, is given in [**5**]: just set $\delta=(1+d_a(x, y)^2/4t)^{-1}$ in formula (I.1.25) of that paper.

The following asymptotic results are corollaries of our global estimates.

COROLLARY 1.3. *Let a and $b-\hat{b}$ be uniformly continuous. Let $\alpha\in(\frac{1}{2}, 1)$ satisfying $\alpha^2/(2\alpha-1)>\lambda^2$ be given. Then for all $\lambda\in[1,\infty)$, $\Lambda\in[0,\infty)$ and all T, R, $\beta\in(0,\infty)$,*

$$(1)\qquad \lim_{M\nearrow\infty}\ \inf_{\substack{0<u-t\leqslant T\\ x,y\in\mathbb{R}^N\\ E(t,x;u,y)\geqslant M}} \frac{\log p(t, x ; u, y) + E(t, x ; u, y)}{E(t, x ; u, y)^{1/(4\alpha-1)}} = 0,$$

$$(2)\qquad \limsup_{M\nearrow\infty}\ \sup_{\substack{0<u-t\leqslant T\\ x,y\in\mathbb{R}^N\\ E(t,x;u,y)\geqslant M}} \frac{\log p(t, x ; u, y) + E(t, x ; u, y)}{\log E(t, x ; u, y)} \leqslant \frac{N}{4\alpha-2},$$

(3) $$\liminf_{(u-t)\nearrow\infty}\ \inf_{|x-y|\leq R} \frac{\log p(t, x ; u, y) + E_\beta(t, x ; u, y)}{u-t} \geq \inf c,$$

(4) $$\limsup_{(u-t)\nearrow\infty}\ \sup_{|x-y|\leq R} \frac{\log p(t, x ; u, y) + E(t, x ; u, y)}{u-t} \leq \sup(\tfrac{1}{4}|b+\hat{b}|_a^2 + c).$$

Proof. (1) Use the lower bound of Theorem 1.1, together with the obvious control of $|E_\beta(t, x ; u, y) - E(t, x ; u, y)|$ deriving from the hypothesis of uniform continuity.

(2) This is immediate from the upper bound of Theorem 1.1.

(3) This follows from (2.3), (3.4) and (3.6) on choosing $\varepsilon = (u-t)^{-\frac{1}{2}}$.

(4) This is immediate from Theorem 1.1 and (2.3).

The class of propagators considered is stable under a number of transformations. This results in some economy of argument.

(1) *Time reversal and L^2-duality*. The heat equation (1.4) yields information on the heat kernel through (1.3). We have also

$$(\hat{P}_{u,t}\psi)(y) = \int_{\mathbb{R}^N} \psi(x) p(t, x ; u, y)\, dx,$$

where $\hat{P}_{u,t}$ is the L^2-dual of $P_{t,u}$. We note from (1.4) that the coefficients of $\hat{P}_{u,t}$ (when time is considered in the opposite sense) are a, $\hat{b}$, b and c. That is, the only change is that the rôles of b and $\hat{b}$ are reversed.

(2) *Scaling*. For $0<\sigma<\infty$ consider the scaled propagator and heat kernel

$$P^\sigma_{t,u} = (M^\sigma)^{-1} P_{\sigma^2 t, \sigma^2 u} M^\sigma \quad \text{where } (M^\sigma\phi)(y) \equiv \phi(y/\sigma).$$

A simple calculation shows that the corresponding operator L^σ has coefficients

$$a(\sigma^2 t, \sigma x), \quad \sigma b(\sigma^2 t, \sigma x), \quad \sigma\hat{b}(\sigma^2 t, \sigma x) \quad \text{and} \quad \sigma^2 c(\sigma^2 t, \sigma x)$$

and that the associated heat kernel is given by

$$p^\sigma(t, x ; u, y) = \sigma^N p(\sigma^2 t, \sigma x ; \sigma^2 u, \sigma y).$$

Note that, so long as $\sigma \in [0, 1]$, the coefficients again satisfy (1.5).

(3) *Conjugation with a potential*. For $\theta \in C_b^1(\mathbb{R}\times\mathbb{R}^N, \mathbb{R})$, let $\theta_s = \theta(s, \cdot)$ and consider the propagator

$$P^\theta_{t,u} = e^{-\theta_t} P_{t,u} e^{\theta_u}.$$

The associated operator L^θ has coefficients

$$a^\theta = a, \quad b^\theta = b + \nabla\theta, \quad \hat{b}^\theta = \hat{b} - \nabla\theta \quad \text{and} \quad c^\theta = c + \frac{\partial\theta}{\partial t} + |\nabla\theta|_a^2 + \langle b - \hat{b}, \nabla\theta\rangle_a,$$

and the corresponding heat kernel is given by

$$p^\theta(t, x ; u, y) = e^{-\theta(t,x)} p(t, x ; u, y) e^{\theta(u,y)}.$$

This transformation will be used in § 2 in order to get the upper bound.

(4) *Moving along a path*. For $\gamma \in C_b^1(\mathbb{R}, \mathbb{R}^N)$ consider the propagator

$$P^\gamma_{t,u} = \tau_t^{-\gamma} P_{t,u} \tau_u^\gamma \quad \text{where } (\tau_u^\gamma\phi)(y) \equiv \phi(y - \gamma(u)).$$

The associated operator L^γ has coefficients

$$a^\gamma(t,x) = a(t, x+\gamma(t)),$$
$$b^\gamma(t,x) = b(t, x+\gamma(t)) - \tfrac{1}{2}a^\gamma(t,x)^{-1}\dot{\gamma}_t,$$
$$\hat{b}^\gamma(t,x) = \hat{b}(t, x+\gamma(t)) + \tfrac{1}{2}a^\gamma(t,x)^{-1}\dot{\gamma}_t,$$
$$c^\gamma(t,x) = c(t, x+\gamma(t))$$

(the drifts $\hat{b}$ and $\hat{b}^\gamma$ are not uniquely determined, but the choice made preserves the notational L^2-symmetry); and the corresponding heat kernel is given by

$$p^\gamma(t, x, ; u, y) = p(t, x+\gamma(t) ; u, y+\gamma(u)).$$

This transformation is, in a sense, dual to the one in (3) and will be used in § 3 when we derive the lower bound.

2. *The upper bound*

Let $P_{t,u}$, with $t<u$, be a propagator with coefficients a, b, $\hat{b}$ and c, as in § 1. Fixing $u \in \mathbb{R}$ and a positive test-function ϕ on $\mathbb{R}^N$, set $\phi_t = P_{t,u}\phi$, with $t<u$, and, for $q \in (1,\infty)$, consider the function

$$G_q(t) = \|\phi_t\|_q = \left(\int_{\mathbb{R}^N} \phi_t^q\, dx\right)^{1/q} \quad (t<u).$$

We compute the derivative of G_q using the heat equation (1.4). This leads immediately to a bound on $\|P_{t,u}\|_{2\to2}$ (that is, the norm of $P_{t,u}$ as an operator from L^2 to L^2). Next we define the energy function $E(t, x ; u, y)$, generalising the usual Gaussian exponential $|y-x|^2/4(u-t)$. We show in Lemma 2.1 that, when the potential θ is 'dominated' by E, $\|P^\theta_{t,u}\|_{2\to2}$ (see § 1, (3)) satisfies the same bound as $\|P_{t,u}\|_{2\to2}$. After combining this fact with an inequality due to Nash (cf. Lemma 2.2), we obtain a system of differential inequalities for G_{2q} in terms of G_q, where $q \in \{2, 4, 8, \ldots\}$, which leads ultimately to a bound on $\|P_{t,u}\|_{2\to\infty}$. A crude Gaussian upper bound follows immediately. The main results, Theorems 2.7 and 2.8, are then obtained using this crude Gaussian bound, Lemma 2.1, and the sharp bounds on $\|P_{t,u}\|_{2\to2}$.

To compute the derivative of G_q, note that

$$G_q'(t) = \left(\int_{\mathbb{R}^N} \phi_t^q\, dx\right)^{(1-q)/q} \int_{\mathbb{R}^N} \phi_t^{q-1}\frac{\partial\phi}{\partial t}\, dx = G_q(t)^{1-q}\left(\phi_t^{q-1}, \frac{\partial\phi}{\partial t}\right),$$

and that, by the heat equation (1.4),

$$\begin{aligned}\left(\phi_t^{q-1}, \frac{\partial\phi}{\partial t}\right) &= (q-1)(\phi_t^{q-2}\nabla\phi_t, \nabla\phi_t)_a - (\phi_t^{q-1}b, \nabla\phi_t)_a \\ &\qquad - (q-1)(\phi_t^{q-2}\nabla\phi_t, \hat{b}\phi_t)_a - (\phi_t^{q-1}, c\phi_t) \\ &= \frac{4(q-1)}{q^2}(\nabla(\phi_t^{q/2}), \nabla(\phi_t^{q/2}))_a \\ &\qquad - 2\left(\nabla(\phi_t^{q/2}), \left(\frac{b}{q} + \left(1-\frac{1}{q}\right)\hat{b}\right)\phi_t^{q/2}\right)_a - (c, \phi_t^q).\end{aligned}$$

Thus, by the quadratic inequality $\pm 2xy \leqslant \delta x^2 + y^2/\delta$, where $0 < \delta < \infty$,

$$(2.1)\quad \left(\phi_t^{q-1}, \frac{\partial\phi}{\partial t}\right) \geqslant \frac{4}{q}\left(1 - \frac{1}{q} - \delta\right)(\nabla(\phi_t^{q/2}), \nabla(\phi_t^{q/2}))_a - \left(\frac{q}{4\delta}\left|\frac{b}{q} + (1 - 1/q)\hat{b}\right|_a^2 + c, \phi_t^q\right).$$

Taking $\delta = 1 - 1/q$, we get

$$G_q'(t) \geqslant - \sup_{x \in \mathbb{R}^N}\left[\frac{|b + (q-1)\hat{b}|_a^2}{4(q-1)} + c(t, x)\right]G_q(t).$$

Since $\phi > 0$ is arbitrary and $|P_{t,u}\psi| \leqslant P_{t,u}|\psi|$ for any test-function ψ, this implies a bound on $P_{t,u}$ as an operator on L^q, where $q \in (1, \infty)$. We note in particular that

$$(2.2)\qquad \|P_{t,u}\|_{2\to 2} \leqslant \exp\left[\int_t^u \sup_{x \in \mathbb{R}^N}\left(\tfrac{1}{4}|b + \hat{b}|_a^2 + c\right)(s, x)\, ds\right].$$

Recall the definition of the energy function E: for $t, u \in \mathbb{R}$ with $t < u$ and $x, y \in \mathbb{R}^N$,

$$E(t, x\,; u, y) = \inf\left\{\frac{1}{4}\int_t^u |\dot{\gamma}_s - a(b - \hat{b})|_{a^{-1}}^2(s, \gamma_s)\, ds:\ \gamma \in C^1([t, u], \mathbb{R}^N) \text{ with } \gamma_t = x \text{ and } \gamma_u = y\right\}.$$

(The restriction from absolutely continuous paths to C^1-paths does not affect the value of the infimum.)

By the nature of its definition it will be very hard to calculate E in general; indeed it is not trivial even to show that E is continuous (see the Appendix). Nevertheless it is clear that one has the estimate

$$(2.3)\qquad \frac{|y - x|^2}{8\lambda(u - t)} - \tfrac{1}{4}\Lambda(u - t) \leqslant E(t, x\,; u, y) \leqslant \frac{\lambda|y - x|^2}{2(u - t)} + \tfrac{1}{2}\Lambda(u - t),$$

in terms of the coefficient bounds λ and Λ. The following result is the first step to getting an upper bound on $p(t, x\,; u, y)$ in terms of $E(t, x\,; u, y)$.

LEMMA 2.1. *Suppose that a and $b - \hat{b}$ are uniformly continuous and that θ is a bounded measurable function on $[t, u] \times \mathbb{R}^N$ satisfying*

$$(2.4)\qquad \theta(s', z') - \theta(s, z) \leqslant E(s, z\,; s', z')$$

whenever $t \leqslant s < s' \leqslant u$ and $z, z' \in \mathbb{R}^N$. Then the propagator $P_{t,u}^\theta = e^{-\theta_t}P_{t,u}e^{\theta_u}$ satisfies

$$\|P_{t,u}^\theta\|_{2\to 2} \leqslant \exp\left[\int_t^u \sup_{z \in \mathbb{R}^N}\left(\tfrac{1}{4}|b + \hat{b}|_a^2 + c\right)(s, z)\, ds\right].$$

Proof. Take $v = \frac{1}{2}(t + u)$ and $s \leqslant v \leqslant s'$. Then, by (2.4), we have

$$\theta(s', z') \leqslant \tilde{\theta}(s', z') \equiv \inf_{w \in \mathbb{R}^N}[\theta(v, w) + E(v, w; s', z')],$$

$$\theta(s, z) \geqslant \bar{\theta}(s, z) \equiv \sup_{w \in \mathbb{R}^N}[\theta(v, w) - E(s, z\,; v, w)].$$

Since $\tilde{\theta}$ also satisfies (2.4), it suffices to prove the lemma for $\tilde{\theta}$. By Theorem A.6 (p. 400) the functions $E(v, w; \cdot, \cdot)$ are equi-continuous in $(v, \infty) \times \mathbb{R}^N$ as w ranges over compacts. Since θ is bounded and

$$E(v, w; s', z') \geq \frac{|z'-w|^2}{8\lambda(s'-v)} - \tfrac{1}{4}\Lambda(s'-v),$$

this implies that $\tilde{\theta}$ is continuous in $(v, \infty) \times \mathbb{R}^N$, and a similar argument shows that it is also continuous on $(-\infty, v) \times \mathbb{R}^N$. It therefore suffices to prove the lemma under the additional hypothesis that θ is continuous away from $\{v\} \times \mathbb{R}^N$.

Take any smooth, compactly supported function $\psi \geq 0$ on $\mathbb{R} \times \mathbb{R}^N$ of integral 1, and set $\psi_n(s, z) = n^{N+1}\psi(ns, nz)$. Consider the approximating sequence $\theta_n = \psi_n * \theta$, where $*$ denotes convolution in $\mathbb{R} \times \mathbb{R}^N$. By (2.4), we have

$$\theta_n(s', z') - \theta_n(s, z) \leq \frac{1}{4}\int_s^{s'} (|\dot{\gamma}_r - a(b-\hat{b})|^2_{a^{-1}} * \psi_n)(r, \gamma_r)\, dr$$

for all paths γ connecting (s, z) and (s', z'). By the uniform continuity of a and $b - \hat{b}$, there is then a sequence $\delta_n \searrow 0$ such that $\tilde{\theta}_n \equiv \theta_n/(1+\delta_n)$ satisfies

$$\tilde{\theta}_n(s', z') - \tilde{\theta}_n(s, z) \leq E(s, z; s', z'). \tag{2.5}$$

Fix (s, z) and consider any path γ with $\gamma_s = z$; then

$$\begin{aligned}
\tilde{\theta}_n(s', \gamma_{s'}) - \tilde{\theta}_n(s, z) &= \int_s^{s'} \frac{d}{dr}(\tilde{\theta}_n(r, \gamma_r))\, dr \\
&= \int_s^{s'} \left(\frac{\partial \tilde{\theta}_n}{\partial r} + \langle \dot{\gamma}_r, \nabla\tilde{\theta}_n\rangle\right)(r, \gamma_r)\, dr \\
&= \int_s^{s'} \left(\frac{\partial \tilde{\theta}_n}{\partial r} + |\nabla\tilde{\theta}_n|^2_a + \langle \nabla\tilde{\theta}_n, b-\hat{b}\rangle_a\right)(r, \gamma_r)\, dr \\
&\quad + \frac{1}{4}\int_s^{s'} |\dot{\gamma}_r - a(b-\hat{b})|^2_{a^{-1}}(r, \gamma_r)\, dr \\
&\quad - \int_s^{s'} |a^{\frac{1}{2}}\nabla\tilde{\theta}_n - \tfrac{1}{2}a^{-\frac{1}{2}}(\dot{\gamma}_r - a(b-\hat{b}))|^2(r, \gamma_r)\, dr.
\end{aligned}$$

By (2.5), the left-hand side is no greater than the second integral on the right. Hence, since we can choose γ to make the third integrand vanish at $r = s$, differentiation leads to

$$\left(\frac{\partial \tilde{\theta}_n}{\partial s} + |\nabla\tilde{\theta}_n|^2_a + \langle \nabla\tilde{\theta}_n, b-\hat{b}\rangle_a\right)(s, z) \leq 0. \tag{2.6}$$

The coefficients of the propagator $P^{\tilde{\theta}_n}_{t,u} \equiv e^{-\tilde{\theta}_{n,t}}P_{t,u}e^{\tilde{\theta}_{n,u}}$ are given in § 1, (3). In particular,

$$b^{\tilde{\theta}_n} + \hat{b}^{\tilde{\theta}_n} = b + \hat{b},$$

and, by (2.6), $c^{\tilde{\theta}_n} \leq c$. Thus (2.2) implies that

$$\|P^{\tilde{\theta}_n}_{t,u}\|_{2\to2} \leq \exp\left[\int_t^u \sup_{z\in\mathbb{R}^N}(\tfrac{1}{4}|b+\hat{b}|^2_a + c)(s, z)\, ds\right].$$

Finally, $\|P^{\tilde{\theta}_n}_{t,u}\|_{2\to2} \to \|P^{\theta}_{t,u}\|_{2\to2}$ by the continuity of θ.

We have not yet exploited the full power of (2.1): the term in $(\nabla(\phi_t^{q/2}), \nabla(\phi_t^{q/2}))$ can be turned to advantage by means of the following fundamental result.

LEMMA 2.2 (Nash's Inequality). *There is a constant $C(N) < \infty$ such that, for all test-functions ϕ on $\mathbb{R}^N$,*

$$\|\phi\|_2^{2+4/N} \leq C \|\nabla\phi\|_2^2 \|\phi\|_1^{4/N}.$$

Proof. Just for now, let P_t be the classical heat semigroup on $\mathbb{R}^N$. Recall that P_t is a self-adjoint contraction on L^2, P_t commutes with ∇, $\|P_t\|_{1\to\infty} = (4\pi t)^{-N/2}$, and

$$P_t\phi = \phi + \int_0^t \Delta P_s\phi\, ds.$$

Thus,

$$(\phi, P_t\phi) = (\phi, \phi) + \int_0^t (\phi, \Delta P_s\phi)\, ds,$$

and so

$$\|\phi\|_2^2 = (\phi, P_t\phi) + \int_0^t (P_{s/2}\nabla\phi, P_{s/2}\nabla\phi)\, ds \leq (4\pi t)^{-N/2}\|\phi\|_1^2 + t\|\nabla\phi\|_2^2.$$

Now optimise over $0 < t < \infty$.

By Nash's inequality and the uniform positivity of a, we have

$$(\nabla(\phi_t^{q/2}), \nabla(\phi_t^{q/2}))_a \geq \frac{1}{\lambda}\|\nabla(\phi_t^{q/2})\|_2^2 \geq \frac{\|\phi_t^{q/2}\|_2^{2+4/N}}{C\lambda\|\phi_t^{q/2}\|_1^{4/N}} = \frac{G_q(t)^{q(1+2/N)}}{C\lambda G_{q/2}(t)^{2q/N}}.$$

Now, by taking $\delta = \frac{1}{2}$ in (2.1), we get

$$G_q'(t) \geq \frac{(2-4/q)}{C\lambda q}\frac{G_q(t)^{1+2q/N}}{G_{q/2}(t)^{2q/N}} - m_q(t)G_q(t), \tag{2.7}$$

where

$$m_q(t) = \sup_{x\in\mathbb{R}^N}\left[\frac{q}{2}\left|\frac{b}{q} + \left(1-\frac{1}{q}\right)\hat{b}\right|_a^2 + c\right](t, x).$$

Now suppose $\|\phi\|_2 = 1$; then (2.2) would give us a bound on $G_2(t)$, for $t < u$. Then (2.7) with $q = 4$ would lead to a bound on $G_4(t)$, from which we would get a bound on $G_8(t)$, and so on. We have to carry out this procedure with some care because we ultimately want to let $q \to \infty$. For this reason, we first show how (2.7) is related to a simpler system of differential equations, and then we apply an elementary estimate on that system to get the desired bounds.

Let μ and $\hat{\mu}$ be uniform bounds on $|b|_a^2$ and $|\hat{b}|_a^2$, respectively. Set $\varepsilon = 2/C\lambda N$ and, for $q = 2, 4, 8, \ldots$, define

$$M_q(t) = (\tfrac{1}{2}q - 1)\hat{\mu}(u-t) + \int_t^u \sup_{x\in\mathbb{R}^N}(\tfrac{1}{4}|b + \hat{b}|_a^2 + c)(s, x)\, ds,$$

$$F_q(t) = \varepsilon[e^{-M_q(t)}G_q(t)]^{-2q/N}.$$

We have

$$m_2(t) = -M_2'(t),$$

$$m_q(t) - m_{q/2}(t) \leq \tfrac{1}{4}q\hat{\mu} = -(M_q - M_{q/2})'(t), \quad \text{for } q = 4, 8, \ldots.$$

Hence,

$$m_q(t) \leqslant -M_q'(t), \quad \text{for } q = 2, 4, 8, \ldots,$$

and therefore, by (2.7),

$$\begin{aligned} F_q'(t) &= -\frac{2q\varepsilon}{N} \exp\left[\frac{2qM_q(t)}{N}\right] G_q(t)^{-(1+2q/N)}[G_q'(t) - M_q'(t)G_q(t)] \\ &\leqslant -\frac{4(q-2)\varepsilon}{qC\lambda N}(e^{-M_q(t)}G_q(t))^{-2q/N}. \end{aligned}$$

When $q = 2$, this only shows that $F_2'(t) \leqslant 0$ and therefore that $F_2(t) \geqslant F_2(u) = \varepsilon$. But, for $q = 4, 8, \ldots$, we get

$$\begin{aligned} F_q'(t) &\leqslant -\varepsilon^2 \exp\left[\frac{2q}{N}(M_q(t) - M_{q/2}(t))\right](e^{-M_{q/2}(t)}G_{q/2}(t))^{-2q/N} \\ &= -e^{\beta q^2(u-t)}(F_{q/2}(t))^2, \end{aligned} \tag{2.8}$$

where $\beta = \hat{\mu}/2N$.

LEMMA 2.3. *Let $\beta \geqslant 0$ and $\varepsilon > 0$. Define inductively a sequence of functions $f_q(t)$, for $0 \leqslant t < \infty$, $q = 2, 4, 8, \ldots$, by*

$$f_2(t) \equiv \varepsilon \quad \textit{and} \quad f_q(t) = \int_0^t e^{\beta q^2 s}(f_{q/2}(s))^2\, ds \ \textit{for } q = 4, 8, \ldots .$$

Then there is an absolute constant $A > 0$ such that

$$f_q(t) \geqslant \varepsilon^{q/2}(Ate^{4\beta(q-1)t})^{q/2-1} \tag{2.9}$$

for all $0 \leqslant t < \infty$ and each q.

Proof. We will define a decreasing sequence A_q for which (2.9) holds for $A = A_q$. It will then suffice to observe that $\lim A_q > 0$. Take $A_2 = 1$. Assuming that we have found $A = A_q \leqslant 1$ such that (2.9) holds, we see that

$$\begin{aligned} f_{2q}(t) &\geqslant \int_{t(1-1/4q^2)}^t e^{4\beta q^2 s}(f_q(s))^2\, ds \\ &\geqslant \frac{t}{4q^2\varepsilon^q}\exp[4\beta(1-1/4q^2)(q^2+(q-1)(q-2))t](At(1-1/4q^2))^{q-2}. \end{aligned}$$

Using the inequality $(1-1/4q^2)(q^2+(q-1)(q-2)) \geqslant (2q-1)(q-1)$ for the exponential term, we see that this implies

$$f_{2q}(t) \geqslant \varepsilon^q[A(1-1/4q^2)(1/4q^2)^{1/(q-1)}te^{\beta(2q-1)t}]^{q-1}.$$

Thus the recursive relation

$$A_{2q} = A_q(1-1/4q^2)(1/4q^2)^{1/(q-1)}$$

defines a suitable sequence.

THEOREM 2.4. *Recall that $\hat{\mu}$ is a uniform bound on $|\hat{b}|_a^2$. There is a constant $C = C(\lambda, N) < \infty$ such that*

$$\|P_{t,u}\|_{2\to\infty} \leqslant \left(\frac{C}{u-t}\right)^{N/4} \exp\left[\tfrac{1}{2}\hat{\mu}(u-t) + \int_t^u \sup_{x\in\mathbb{R}^N}(\tfrac{1}{4}|b+\hat{b}|_a^2 + c)(s,x)\, ds\right].$$

Proof. After reversing the direction of time, apply Lemma 2.3 to (2.8) to obtain

$$F_q(t) \geqslant \varepsilon^{q/2}(A(u-t)e^{4\beta(q-1)(u-t)})^{q/2-1}$$

with $\beta = \hat{\mu}/2N$. Thus, for $\|\phi\|_2 = 1$,

$$\|\phi_t\|_q = G_q(t) = e^{M_q(t)}(F_q(t)/\varepsilon)^{-N/2q}$$

$$\leqslant \left(\frac{1}{A\varepsilon(u-t)}\right)^{N(q-2)/4q} \exp\left[\tfrac{1}{2}\hat{\mu}(1-2/q)(u-t) + \int_t^u \sup_{x\in\mathbb{R}^N} (\tfrac{1}{4}|b+\hat{b}|_a^2 + c)(s,x)\,ds\right];$$

and so the theorem follows from

$$\|P_{t,u}\|_{2\to\infty} \leqslant \overline{\lim_{q\to\infty}} \|P_{t,u}\|_{2\to q}.$$

COROLLARY 2.5. *There is a constant $C(\lambda, N) \in (0,\infty)$ such that*

$$p(t,x\,;u,y) \leqslant \frac{C}{(u-t)^{N/2}} \exp\left[C\Lambda(u-t) - \frac{|x-y|^2}{C\lambda(u-t)}\right]$$

for all $t<u$ and $x, y \in \mathbb{R}^N$.

Proof. Fix $x, y \in \mathbb{R}^N$, $0<\alpha<\infty$, and set

$$\theta(z) = \alpha(|x-z| \wedge |x-y|), \quad \text{where } z \in \mathbb{R}^N.$$

Recalling that the propagator $P^\theta_{t,u} = e^{-\theta_t}P_{t,u}e^{\theta_u}$ corresponds to coefficients $b^\theta = b + \nabla\theta$, $\hat{b}^\theta = \hat{b} - \nabla\theta$, and

$$c^\theta = c + |\nabla\theta|_a^2 + \langle\nabla\theta, b-\hat{b}\rangle_a,$$

apply Theorem 2.4 and conclude that, for some $C(\lambda, N) \in (0,\infty)$,

$$\|P^\theta_{t,v}\|_{2\to\infty} \leqslant \left(\frac{2C}{u-t}\right)^{N/4} \exp[\tfrac{1}{8}C\alpha^2\lambda(u-t) + \tfrac{1}{2}C\Lambda(u-t)] \quad \text{where } v = \tfrac{1}{2}(t+u).$$

Hence, since

$$\|P^\theta_{v,u}\|_{1\to 2} = \|\hat{P}^\theta_{u,v}\|_{2\to\infty},$$

which satisfies a similar bound, we have

$$p(t,x\,;u,y) = e^{-\alpha|x-y|}p^\theta(t,x\,;u,y) \leqslant e^{-\alpha|x-y|}\,\|P^\theta_{t,v}\|_{2\to\infty}\,\|P^\theta_{v,u}\|_{1\to 2}$$

$$\leqslant \left(\frac{2C}{u-t}\right)^{N/2} \exp[\tfrac{1}{4}C\alpha^2\lambda(u-t) + C\Lambda(u-t) - \alpha|x-y|].$$

Now optimise the choice of α.

LEMMA 2.6. *Assume a and $b-\hat{b}$ are uniformly continuous. Let $(u,y) \in (0,\infty) \times \mathbb{R}^N$ be given. There is a bounded measurable function θ: $[0,u] \times \mathbb{R}^N \to \mathbb{R}$ such that*

(2.10) $$\theta(u,y) - \theta(0,0) = E(0,0\,;u,y),$$

and

(2.11) $\theta(s',z') - \theta(s,z) \leqslant E(s,z\,;s',z')$, *for* $0\leqslant s<s'\leqslant u$ *and* $z, z' \in \mathbb{R}^N$.

Moreover, θ may be chosen so that, for each $\alpha \in (\frac{1}{2}, 1)$ satisfying $\alpha^2/(2\alpha-1) > \lambda^2$, there is a $C = C(\alpha, \lambda) \in (0, \infty)$ such that, setting $K = C(|y| + u)^{1/\alpha}/u$, we have

(2.12) $\theta(0, z) - \theta(0, 0) \leqslant K\,|z|^{2-1/\alpha}$ *and* $\theta(u, y) - \theta(u, z) \leqslant K\,|z-y|^{2-1/\alpha}$

for all $z \in \mathbb{R}^N$.

Proof. We will assume $t = 0$ and $x = 0$. Take a path $\gamma^* \in \Gamma(0, 0; u, y)$ of minimal energy, set $v = \frac{1}{2}u$ and $w^* = \gamma_v^*$. Then

$$E(0, 0; v, w^*) + E(v, w^*; u, y) \leqslant E(0, 0; v, w) + E(v, w; u, y) \quad \text{for all } w \in \mathbb{R}^N,$$

so there is a function θ_v on $\mathbb{R}^N$ such that

$$E(v, w^*; u, y) - E(v, w; u, y) \leqslant \theta_v(w) \leqslant E(0, 0; v, w) - E(0, 0; v, w^*) \quad \text{for all } w \in \mathbb{R}^N,$$

and

$$-E(0, 0; v, w^*) \leqslant \theta_v \leqslant E(v, w^*; u, y) \quad \text{uniformly.}$$

Choose such a θ_v and define

$$\theta(s, z) = \begin{cases} \sup_{w \in \mathbb{R}^N} [\theta_v(w) - E(s, z; v, w)] & \text{if } 0 \leqslant s \leqslant v, \\ \inf_{w \in \mathbb{R}^N} [\theta_v(w) + E(v, w; s, z)] & \text{if } v \leqslant s \leqslant u. \end{cases}$$

It is a simple matter to check that θ satisfies (2.10) and (2.11). To show that θ satisfies (2.12) we use the following fact proved in the Appendix, Theorem A.6: *there is a constant $C(\alpha, \lambda) < \infty$ such that, for all $w, z \in \mathbb{R}^N$,*

(2.13) $E(0, z; v, w) - E(0, 0; v, w) \leqslant (C/v)[|w| + |w - z| + \Lambda^{\frac{1}{2}}v]^{1/\alpha}\,|z|^{2-1/\alpha}.$

(*Warning*: the Appendix uses different notation, set out at the beginning of the Appendix.) Now consider two cases. First suppose $|z| \geqslant |y| + \Lambda^{\frac{1}{2}}u$. Then, for $C = \frac{1}{2}\lambda$,

$$\theta(0, z) - \theta(0, 0) \leqslant E(0, 0; u, y) \leqslant C(|y|^2/u + \Lambda u) \leqslant (C/u)(|y| + \Lambda^{\frac{1}{2}}u)^{1/\alpha}\,|z|^{2-1/\alpha}.$$

Secondly, if $|z| \leqslant |y| + \Lambda^{\frac{1}{2}}u$, then by our crude bounds (2.3) on E, there is a constant $C(\lambda) < \infty$ such that

$$\theta(0, z) = \sup\{\theta_v(w) - E(0, z; v, w) \colon |w| + |z - w| \leqslant C(|y| + \Lambda^{\frac{1}{2}}u)\};$$

hence (2.13) implies for some $C(\alpha, \lambda) < \infty$ that

$$\theta(0, z) - \theta(0, 0) \leqslant (C/u)(|y| + \Lambda^{\frac{1}{2}}u)^{1/\alpha}\,|z|^{2-1/\alpha}.$$

We have established the first of the inequalities (2.12); the second follows by symmetry.

THEOREM 2.7 (the upper bound). *Suppose that a and $b - \hat{b}$ are uniformly continuous. Then, for every $\alpha \in (\frac{1}{2}, 1)$ satisfying*

$$\alpha^2/(2\alpha - 1) > \lambda^2,$$

there is a constant $C(\alpha, \lambda, N) \in (0, \infty)$ such that, for all $t < u$ and $x, y \in \mathbb{R}^N$,

$$p(t, x; u, y) \leqslant \left(\frac{C[1 + \Lambda(u - t) + E(t, x; u, y)]^{1/(2\alpha-1)}}{u - t}\right)^{N/2} \times \exp\left\{\int_t^u \sup_{z \in \mathbb{R}^N} (\tfrac{1}{4}|b + \hat{b}|_a^2 + c)(s, z)\,ds - E(t, x; u, y)\right\}.$$

Proof. We will assume that $t=0$ and $x=0$. Let θ be the function of Lemma 2.6. Define the new propagator $P^{\theta}_{s,s'}=e^{-\theta_s}P_{s,s'}e^{\theta_{s'}}$ as in § 1, (3) and let p^{θ} be its heat kernel. Take $\varepsilon=[2+\Lambda u+E(0,0;u,y)]^{-1/(2\alpha-1)}\in(0,\frac{1}{2})$ and set $t'=\varepsilon u$, $u'=(1-\varepsilon)u$. We have by (2.10),

$$\begin{aligned} e^{E(0,0;u,y)}p(0,0;u,y) &= p^{\theta}(0,0;u,y) \\ &= \iint p^{\theta}(0,0;t',z)p^{\theta}(t',z;u',w)p^{\theta}(u',w;u,y)\,dz\,dw \\ &\leq \|p^{\theta}(0,0;t',\cdot)\|_2\,\|P^{\theta}_{t',u'}\|_{2\to2}\,\|p^{\theta}(u',\cdot\,;u,y)\|_2. \end{aligned}$$

By (2.11) and Lemma 2.1,

$$\|P^{\theta}_{t',u'}\|_{2\to2}\leq \exp\left\{\int_{t'}^{u'}\sup_{z\in\mathbb{R}^N}(\tfrac{1}{4}|b+\hat{b}|_a^2+c)(s,z)\,ds\right\}.$$

By (2.11) again,

$$\theta(t',z)-\theta(0,z)\leq\Lambda t',\quad \text{for all } z\in\mathbb{R}^N,$$

so from (2.12),

$$\theta(t',z)-\theta(0,0)\leq K|z|^{2-1/\alpha}+\Lambda t'.$$

By Corollary 2.5 there is a constant $C(\lambda,N)\in(0,\infty)$ such that

$$p(0,0;t',z)\leq(C/t')^{N/2}e^{C\Lambda t'-|z|^2/Ct'},\quad \text{for all } z\in\mathbb{R}^N.$$

Hence

$$p^{\theta}(0,0;t',z)=e^{\theta(t',z)-\theta(0,0)}p(0,0;t',z)\leq(C/t')^{N/2}e^{(C+1)\Lambda t'}e^{K|z|^{2-1/\alpha}-|z|^2/Ct'},$$

and so there is a constant $C(\alpha,\lambda,N)\in(0,\infty)$ such that

$$\|p^{\theta}(0,0;t',\cdot)\|_2\leq\left(\frac{C}{t'}\right)^{N/4}e^{C\Lambda t'}e^{CK^2 t'^{2-1/\alpha}}\leq\left(\frac{C[1+\Lambda u+E(0,0;u,y)]^{1/(2\alpha-1)}}{u}\right)^{N/4}.$$

By symmetry the same bound applies to $\|p^{\theta}(u',\cdot\,;u,y)\|_2$, and the result follows.

THEOREM 2.8 (the upper bound for Hölder continuous coefficients). *Suppose that a and $b-\hat{b}$ are uniformly continuous. Suppose further that a and $(b-\hat{b})/\Lambda^{\frac{1}{2}}$ are Hölder continuous in time of exponent $\alpha\in(0,1]$ and constant $A\in[0,\infty)$, and that $|b-\hat{b}|_a^2/\Lambda$ is Hölder continuous of exponent α and constant $A/\Lambda^{\alpha/2}$. Then there is a constant $C(\alpha,\lambda,N)\in(0,\infty)$ such that, for all $t,u\in\mathbb{R}$ with $t<u$ and all $x,y\in\mathbb{R}^N$,*

$$\begin{aligned} p(t,x;u,y) &\leq \left(\frac{C[(1+A^{1/\alpha}(u-t))(1+\Lambda(u-t)+E(t,x;u,y))]}{u-t}\right)^{N/2} \\ &\quad\times\exp\left\{\int_t^u\sup_{z\in\mathbb{R}^N}(\tfrac{1}{4}|b+\hat{b}|_a^2+c)(s,z)\,ds-E(t,x;u,y)\right\}. \end{aligned}$$

Proof. We follow the proofs of Lemma 2.6 and Theorem 2.7, the only difference being that by our stronger continuity hypothesis on a and $b-\hat{b}$, we can show that E and hence θ (as constructed in Lemma 2.6) are Lipschitz, with control of the constant. By Theorem A.7, there is a constant $C(\alpha,\lambda)<\infty$ such that

$$E(0, z ; v, w) - E(0, 0 ; v, w) \leq (C/v)[(|w| + |w - z| + \Lambda^{\frac{1}{2}}v)(1 + A^{1/\alpha}v)^{\frac{1}{2}}]\,|z|.$$

Using this bound in place of (2.13), we can replace (2.12) by the following stronger estimates: *there is a constant* $C(\alpha, \lambda) < \infty$ *such that, setting* $K = (C/u)[(|y| + \Lambda^{\frac{1}{2}}u)(1 + A^{1/\alpha}u)^{\frac{1}{2}}]$, *we have, for all* $z \in \mathbb{R}^N$,

$$\theta(0, z) - \theta(0, 0) \leq K\,|z|, \quad \theta(u, y) - \theta(u, z) \leq K\,|z - y|.$$

Take

$$\varepsilon = \tfrac{1}{2}[(1 + A^{1/\alpha}u)(1 + \Lambda u + E(0, 0 ; u, y))]^{-1};$$

then $\varepsilon \in (0, \frac{1}{2})$ and $\varepsilon u K^2 \leq C^2$, so by the argument of Theorem 2.7, there is a constant $C(\lambda, N) < \infty$ such that

$$\|p^{\theta}(0, 0 ; \varepsilon u, \cdot)\|_2 \leq \left(\frac{C[(1 + A^{1/\alpha}u)(1 + \Lambda u + E(0, 0 ; u, y))]}{u}\right)^{N/4}.$$

The rest of the proof is identical to that of Theorem 2.7.

3. *The lower bound*

Let $P_{t,u}$, for $t < u$, be a propagator with coefficients a, b, $\hat{b}$ and c as in § 2. As before, after fixing $u \in \mathbb{R}$ and a test-function $\phi > 0$ on $\mathbb{R}^N$, we set $\phi_t = P_{t,u}\phi$, for $t < u$. Let ρ be a probability density function on $\mathbb{R}^N$. The lower bound is obtained via differential inequalities for functions of the form

$$G(t) = \int_{\mathbb{R}^N} \rho \log \phi_t \, dy,$$

which are derived from the heat equation (1.4). The main argument draws on three other results, firstly a spectral gap estimate for the classical Laplacian (Lemma 3.1), secondly a control on the loss of mass under $P_{t,u}$ (Lemma 3.2), and thirdly the crude *upper* bound of Corollary 2.5. In Lemma 3.3 we choose ρ to be the classical heat kernel and obtain a lower bound for the associated function G. (The spectral gap estimate plays a rôle here analogous to Nash's inequality in § 2, being the way in which information on the classical heat kernel is fed into the argument.) This is then used to get a crude Gaussian lower bound in Theorem 3.4. This first lower bound actually enables us to strengthen Lemma 3.3, in particular to allow a more general choice of the function ρ. Then, using a more careful version of the argument for Theorem 3.4, we get a more precise lower bound in Theorem 3.5. When the coefficients are Hölder continuous this bound may be sharpened further (Theorem 3.7).

We use the following notation:

$$\langle \phi \rangle_\rho = \int_{\mathbb{R}^N} \phi\rho\, dx, \quad (\psi, \phi)_\rho = \int_{\mathbb{R}^N} \psi\phi\rho\, dx, \quad \text{and} \quad \mathrm{var}_\rho(\phi) = \int_{\mathbb{R}^N} (\phi - \langle \phi \rangle_\rho)^2 \rho\, dy.$$

LEMMA 3.1 (spectral gap estimate). *Let* ρ^c_τ, *with* $0 < \tau < \infty$, *be the classical heat kernel*

$$\rho^c_\tau(x) = (4\pi\tau)^{-N/2} \exp(-|x|^2/4\tau), \quad \textit{where } x \in \mathbb{R}^N.$$

Then for any test-function ψ *on* $\mathbb{R}^N$,

$$\mathrm{var}_{\rho^c_\tau}(\psi) \leq 2\tau \langle |\nabla\psi|^2 \rangle_{\rho^c_\tau}.$$

Proof. In the proof we shall suppress the superscript c. By substitution and scaling, we reduce to the case when $\tau = \frac{1}{2}$ and $\langle \psi \rangle_{\rho_{\frac{1}{2}}} = 0$. Set $\rho = \rho_{\frac{1}{2}}$. Let P_t be the Ornstein–Uhlenbeck semi-group on $\mathbb{R}^N$. That is,

$$(P_t\psi)(x) = \int_{\mathbb{R}^N} \psi(y)\rho_{(1-e^{-2t})/2}(y - e^{-t}x)\, dy.$$

Then

$$\frac{\partial}{\partial t} P_t = (\Delta - x \cdot \nabla)P_t \quad \text{and} \quad \nabla \circ P_t = e^{-t}P_t \circ \nabla,$$

and so, by an integration by parts, for $\psi_t = P_t\psi$ we have,

$$\frac{d}{dt}\int_{\mathbb{R}^N} \psi_t^2 \rho\, dy = -2\int_{\mathbb{R}^N} |\nabla\psi_t|^2 \rho\, dy = -2e^{-2t}\int_{\mathbb{R}^N} |P_t\, \nabla\psi|^2 \rho\, dy.$$

Hence, since, as the preceding makes clear,

$$\frac{d}{dt}\int_{\mathbb{R}^N} |P_t\, \nabla\psi|^2 \rho\, dy \leqslant 0,$$

we find that

$$\frac{d}{dt}\int_{\mathbb{R}^N} \psi_t^2 \rho\, dy \geqslant -2e^{-2t}\int_{\mathbb{R}^N} |\nabla\psi|^2 \rho\, dy,$$

which, after integration, yields

$$\lim_{t\to\infty}\int_{\mathbb{R}^N} \psi_t^2 \rho\, dy - \int_{\mathbb{R}^N} \psi^2 \rho\, dy \geqslant -\int_0^\infty 2e^{-2t}\, dt \int_{\mathbb{R}^N} |\nabla\psi|^2 \rho\, dy.$$

Finally, because $\psi_t \to \langle \psi \rangle_\rho = 0$ uniformly on $\mathbb{R}^N$ as $t \to \infty$, we conclude that

$$\int_{\mathbb{R}^N} \psi^2 \rho\, dy \leqslant \int_{\mathbb{R}^N} |\nabla\psi|^2 \rho\, dy.$$

LEMMA 3.2 (loss of mass). *Suppose that* $\Lambda \leqslant 1$ *and* $0 < u - t \leqslant 1$. *Then there is a constant* $0 < C(\lambda, N) < \infty$ *such that*

$$\int_{\mathbb{R}^N} p(t, x\,; u, y)\, dx \geqslant \frac{1}{C} \quad \textit{for all } y \in \mathbb{R}^N.$$

Proof. Obviously, it suffices to handle the case when $y = 0$. To this end, define

$$\omega(x) = \exp[-C(1 + |x|^2)^{\frac{1}{2}}], \quad \text{for } x \in \mathbb{R}^N,$$

where $C = C(N)$ is chosen so that $\int_{\mathbb{R}^N} \omega\, dx = 1$. Clearly, $|\nabla(\log \omega)| \leqslant C$. Next, take any strictly positive test-function ϕ on $\mathbb{R}^N$ with $\int_{\mathbb{R}^N} \phi\, dx = 1$, and set $\phi_t = P_{t,u}\phi$, for $t < u$, and

$$H_\beta(t) = \int_{\mathbb{R}^N} \omega\phi_t^\beta\, dy \quad \text{for } \beta \in (0, 2) \text{ and } t < u.$$

One then has that

$$\begin{aligned} H_\beta'(t) &= \beta\left(\omega\phi_t^{\beta-1}, \frac{d}{dt}\phi_t\right) \\ &= -\beta(1-\beta)(\phi_t^{\beta/2-1}\nabla\phi_t, \phi_t^{\beta/2-1}\nabla\phi_t)_{\omega a} \\ &\quad + \beta(\phi_t^{\beta/2}[\nabla(\log \omega) - b + (1-\beta)\hat{b}], \phi_t^{\beta/2-1}\nabla\phi_t)_{\omega a} \\ &\quad - \beta(\langle \nabla(\log \omega), \hat{b} \rangle_a + c, \phi_t^\beta)_\omega. \end{aligned}$$

By quadratic inequalities we find there is a constant $C(\lambda, N) < \infty$ such that for $0 < \beta < 1$,

$$H'_\beta(t) \leq -\tfrac{1}{2}\beta(1-\beta)(\nabla\phi_t, \phi_t^{\beta-2}\nabla\phi_t)_{a\rho} + \frac{C}{1-\beta}H_\beta(t),$$

and

$$H'_1(t) \leq \tfrac{1}{2}\beta(1-\beta)(\nabla\phi_t, \phi_t^{\beta-2}\nabla\phi_t)_{a\rho} + \frac{C}{1-\beta}H_{2-\beta}(t) + CH_1(t).$$

By the upper bound of Corollary 2.5, there is a constant $C(\lambda, N) < \infty$ such that, for $0 < \beta < 1$,

$$H_{2-\beta}(t) \leq \left(\frac{C}{(u-t)^{N/2}}\right)^{1-\beta} H_1(t).$$

Thus, by taking $\beta = 1 - 1/N$, we get

$$(H_\beta + H_1)'(t) \leq C(1 + (u-t)^{-\frac{1}{2}})(H_\beta + H_1)(t);$$

and therefore, since $H(u) = \langle \phi \rangle_\omega$ and $H_\beta(t) \leq H_1(t)^\beta$, we conclude that

$$H_1(t) \geq C \min\{1, \langle \phi \rangle_\omega^{N/(N-1)}\}.$$

Hence, after letting ϕ tend to the delta function, one has

$$\int_{\mathbb{R}^N} p(t, x\,; u, 0)\, dx \geq C.$$

We return now to the main argument: ρ is a probability density function on $\mathbb{R}^N$, $\phi > 0$ is a test-function on $\mathbb{R}^N$ and $\phi_t = P_{t,u}\phi$, where $t < u$. Setting

$$G(t) = \int_{\mathbb{R}^N} \rho \log \phi_t\, dx,$$

we have, from the heat equation (1.4),

$$\begin{aligned} G'(t) &= \left(\frac{\rho}{\phi_t}, \frac{d}{dt}\phi_t\right) \\ &= -(\nabla(\log\phi_t), \nabla(\log\phi_t))_{a\rho} - (\nabla(\log\phi_t), b - \hat{b} - \nabla(\log\rho))_{\rho a} \\ &\quad -(\nabla(\log\rho), \hat{b})_{\rho a} - \langle c \rangle_\rho. \end{aligned}$$

Fix some probability density function ρ_1 with the property that

$$H = H(\rho_1) \equiv \int_{\mathbb{R}^N} \frac{|\nabla\rho_1|^2}{\rho_1}\, dx < \infty.$$

Define, for $0 < \tau < \infty$,

$$\rho_\tau(x) = \tau^{-N/2}\rho_1(\tau^{-\frac{1}{2}}x), \quad \text{where } x \in \mathbb{R}^N;$$

then $H(\rho_\tau) = H/\tau$. If we take $\rho = \rho_\tau$ above and write $G = G_\tau$, then on completing squares in various ways, we find both that

$$G'_\tau(t) \leq -\frac{1}{4}\int_{\mathbb{R}^N} \rho_\tau\, |\nabla(\log\phi_t)|_a^2\, dx + C(\lambda, N, \Lambda, \tau, H), \tag{3.1}$$

and that for all $\varepsilon' \in (0, \infty)$,

$$G'_\tau(t) \leq \frac{C(\lambda, N, H)}{\tau}\left(1 + \frac{1}{\varepsilon'}\right) + \int_{\mathbb{R}^N} \rho_\tau\{\tfrac{1}{4}|b - \hat{b}|_a^2 + \varepsilon'\,|b + \hat{b}|_a^2 - c\}\, dx. \tag{3.2}$$

Both these inequalities will be used below: (3.1) will be applied in the following lemma in the case where $\tau = 1$ and

$$\rho_1(x) = (4\pi)^{-N/2} \exp(-\tfrac{1}{4}|x|^2), \quad \text{for } x \in \mathbb{R}^N,$$

when we have $H(\rho_1) = \frac{1}{2}N$. We will apply (3.2) with ρ_τ Gaussian in Theorem 3.4, and more generally in Theorem 3.5. The inequalities lie at the heart of our method and have been derived together at this stage to emphasise the unity of the techniques involved.

LEMMA 3.3. *Let ρ_τ^c, for $0 < \tau < \infty$, be the classical heat kernel*

$$\rho_\tau^c(x) = (4\pi\tau)^{-N/2} \exp(-|x|^2/4\tau), \quad \textit{where } x \in \mathbb{R}^N.$$

Assume that $\Lambda \leqslant 1$ and $0 < u - t \leqslant 1$. Then there is a constant $C(\lambda, N) < \infty$ such that

$$\int_{\mathbb{R}^N} \rho_{u-t}^c(x) \log p(t, x; u, 0)\, dx \geqslant -\tfrac{1}{2}N \log(u-t) - C. \tag{3.3}$$

Proof. In the proof we shall suppress the superscript c. By the scaling transformation of § 1, (2) we see that, for $0 < \sigma < \infty$, (3.3) is equivalent to

$$\int_{\mathbb{R}^N} \rho_{\sigma^2(u-t)}(x') \log p'(\sigma^2 t, x', \sigma^2 u, 0)\, dx' \geqslant -\tfrac{1}{2}N \log \sigma^2(u-t) - C,$$

where p' is a heat kernel corresponding to coefficients with $\lambda' = \lambda$ and $\Lambda' = \Lambda/\sigma^2$. In particular, choosing $\sigma^2 = 1/(u-t)$, we still have that $\Lambda' \leqslant 1$ and are thereby reduced to the case where $u - t = 1$. By Corollary 2.5 and Lemma 3.2, we know that there is a $C = C(\lambda, N) \in (0, \infty)$ such that

$$p(t, x; u, y) \leqslant \frac{C}{(u-t)^{N/2}} \exp[-|y-x|^2/C(u-t)], \quad \text{for } 0 < u - t \leqslant 1 \text{ and } x, y \in \mathbb{R}^N,$$

and

$$\int_{\mathbb{R}^N} p(t, x; u, y)\, dx \geqslant \frac{1}{C} \quad \text{for } 0 < u - t \leqslant 1 \text{ and } y \in \mathbb{R}^N.$$

Thus, for some $R = R(\lambda, N) \in (0, \infty)$,

$$\int_{|x| \geqslant R} p(t, x; u, y)\, dx \leqslant \frac{1}{2C} \quad \text{for } 0 < u - t \leqslant 1 \text{ and } |y| \leqslant 1.$$

Now let $\phi > 0$ be any test function with $\int_{\mathbb{R}^N} \phi(y)\, dy = 1$, $\int_{|y| \leqslant 1} \phi(y)\, dy \geqslant \frac{1}{2}$ and define

$$G(t) = G_1(t) = \int_{\mathbb{R}^N} \rho_1(x) \log \phi_t(x)\, dx, \quad \text{for } t < u,$$

as above. Then by Lemma 3.1 we have, on the one hand, that

$$\begin{aligned}\int_{\mathbb{R}^N} \rho_1 |\nabla(\log \phi_t)|_a^2\, dx &\geqslant \frac{1}{2\lambda} \int_{\mathbb{R}^N} \rho_1 (\log \phi_t - G(t))^2\, dx \\ &\geqslant \frac{e^{-R^2/4}}{2\lambda(4\pi)^{N/2}} (-K - G(t))^2 \operatorname{vol}(\{x\colon |x| \leqslant R \text{ and } \phi_t(x) \geqslant e^{-K}\}).\end{aligned}$$

On the other hand, by the preceding paragraph, we know that

$$\phi_t \leq 2^{N/2}C \text{ for } \tfrac{1}{2} \leq u-t \leq 1 \quad \text{and} \quad \int_{|x| \leq R} \phi_t(x)\,dx \geq \frac{1}{4C} \quad \text{for } 0<u-t \leq 1,$$

and therefore that

$$\frac{1}{4C} \leq \int_{|x|\leq 1} \phi_t(x)\,dx \leq e^{-K}\operatorname{vol}(\{x\colon\ |x| \leq R\})$$
$$+ C\operatorname{vol}(\{x\colon\ |x| \leq R \text{ and } \phi_t(x) \geq e^{-K}\}), \quad \text{for } \tfrac{1}{2} \leq u-t \leq 1.$$

Hence, there is choice of $K = K(\lambda, N) \in (0, \infty)$ for which

$$\operatorname{vol}(\{x\colon\ |x| \leq R \text{ and } \phi_t(x) \geq e^{-K}\}) \geq 1/K \quad \text{for } \tfrac{1}{2} \leq u-t \leq 1,$$

and with this choice we have by (3.1) that

$$G'(t) \leq -\varepsilon(-K - G(t))^2 + M, \quad \text{for } \tfrac{1}{2} \leq u-t \leq 1,$$

for some $\varepsilon = \varepsilon(\lambda, N) > 0$ and $M = M(\lambda, N) \in (0, \infty)$. Thus, if $G(u-1) \leq -2K - \frac{1}{2}M$, then $G(t) \leq 2K$, $\frac{1}{2} \leq u-t \leq 1$, and so

$$G'(t) \leq -\tfrac{1}{4}\varepsilon G(t)^2 + M \quad \text{for } \tfrac{1}{2} \leq u-t \leq 1,$$

from which it is easy to deduce that

$$G(u-1) \geq -\left[(2K + \tfrac{1}{2}M) \vee \left(\frac{M}{\varepsilon}\tanh^{-1}(\tfrac{1}{2}(\varepsilon M)^{\frac{1}{2}})\right)\right].$$

Finally by making ϕ tend to the delta function at 0, we find that

$$\int_{\mathbb{R}^N} \rho_1(x) \log p(u-1, x\,; u, 0)\,dx \geq -C(\Lambda, N),$$

as required.

THEOREM 3.4. *There is a constant* $0 < C(\lambda, N) < \infty$ *such that, for all* $0 \leq \Lambda < \infty$, $t < u$ *and* $x, y \in \mathbb{R}^N$,

$$p(t, x\,; u, y) \geq \left(\frac{1}{C(u-t)}\right)^{N/2} \exp\left[-C\Lambda(u-t) - \frac{C\,|y-x|^2}{u-t}\right].$$

Proof. We assume that $t = 0$ and $x = 0$, and initially that $u \leq 1$ and $\Lambda \leq 1$. Define a path $\gamma\colon\ [0, u] \to \mathbb{R}^N$, by

$$\gamma(s) = \begin{cases} 0, & \text{for } 0 \leq s \leq \frac{1}{4}u, \\ 2\left(\dfrac{s}{u} - \dfrac{1}{4}\right)y, & \text{for } \frac{1}{4}u \leq s \leq \frac{3}{4}u, \\ y, & \text{for } \frac{3}{4}u \leq s \leq u. \end{cases}$$

Consider the new heat kernel

$$p^{\gamma}(s, z\,; s', z') = p(s, z + \gamma(s)\,; s', z' + \gamma(s')).$$

The coefficients associated with p^{γ} are listed in § 1, (4). By our choice of γ, for $0 \leq s \leq \frac{1}{4}u$, we have $\dot{\gamma}_s = 0$, so $\Lambda^{\gamma} \equiv |b_s^{\gamma}|_a^2 + |\hat{b}_s^{\gamma}|_a^2 + |c_s^{\gamma}| \leq 1$. Let ρ_{τ}^c, for $0 < \tau < \infty$, be the classical heat kernel as in Lemma 3.3. For the remainder of the proof we suppress the superscript c. By Lemma 3.3,

$$\int_{\mathbb{R}^N} \rho_{u/4}(z)\log p^{\gamma}(0, 0; \tfrac{1}{4}u, z)\, dz \geq -\tfrac{1}{2}N \log \tfrac{1}{4}u - C.$$

On the other hand, (3.2) shows, on taking $\varepsilon' = 1$ and letting ϕ tend to a delta function at 0, that

$$\frac{d}{ds}\int_{\mathbb{R}^N} \rho_{u/4}(z)\log p^{\gamma}(0, 0; s, z)\, dz \geq -\frac{4C}{u} - 2 - \tfrac{1}{4}(\rho_{u/4} * |\dot{\gamma}_s - a(b - \hat{b})|^2_{a^{-1}})(s, \gamma_s).$$

Integrating this inequality from $s = \frac{1}{4}u$ to $s = \frac{1}{2}u$ and adding the result to the one above we get

$$\begin{aligned}\int_{\mathbb{R}^N} &\rho_{u/4}(z)\log p^{\gamma}(0, 0; \tfrac{1}{2}u, z)\, dz \\ &\geq -\tfrac{1}{2}N \log \tfrac{1}{4}u - C - C - \tfrac{1}{2}u - \frac{1}{4}\int_{u/4}^{u/2} \rho_{u/4} * |\dot{\gamma}_s - a(b - \hat{b})|^2_{a^{-1}}(s, \gamma_s)\, ds \\ &\geq -\tfrac{1}{2}N \log Cu - C\,|y|^2/u,\end{aligned}$$

for some $0 < C(\lambda, N) < \infty$. Similarly,

$$\int_{\mathbb{R}^N} \rho_{u/4}(z)\log p^{\gamma}(\tfrac{1}{2}u, z; u, 0)\, dz \geq -\tfrac{1}{2}N \log Cu - C\,|y|^2/u.$$

Hence, by Jensen's inequality,

$$\begin{aligned}\log p(0, 0; u, y) &= \log p^{\gamma}(0, 0; u, 0) \\ &= \log\left[\int_{\mathbb{R}^N} p^{\gamma}(0, 0; \tfrac{1}{2}u, z)p^{\gamma}(\tfrac{1}{2}u, z; u, 0)\, dz\right] \\ &\geq \log\left[\int_{\mathbb{R}^N} p^{\gamma}(0, 0; \tfrac{1}{2}u, z)p^{\gamma}(\tfrac{1}{2}u, z; u, 0)\frac{\rho_{u/4}(z)}{\|\rho_{u/4}\|_\infty}\, dz\right] \\ &\geq \tfrac{1}{2}N \log \pi u + \int_{\mathbb{R}^N} \rho_{u/4}(z)\log p^{\gamma}(0, 0; \tfrac{1}{2}u, z)\, dz \\ &\quad + \int_{\mathbb{R}^N} \rho_{u/4}(z)\log p^{\gamma}(\tfrac{1}{2}u, z; u, 0)\, dz \\ &\geq -\tfrac{1}{2}N \log Cu - C\,|y|^2/u.\end{aligned}$$

In the case where $0 < u - t \leq 1$ and $\Lambda \leq 1$ we have shown that

$$p(t, x; u, y) \geq \left(\frac{1}{C(u - t)}\right)^{N/2} e^{-C|y - x|^2/(u - t)}.$$

To remove the restriction that $u - t \leq 1$, we use

$$p(t, x; u, y) = \int_{\mathbb{R}^N} p(t, x; v, z)p(v, z; u, y)\, dz$$

for $t < v < u$, together with the corresponding property of the classical heat kernel, in order to show that the preceding estimate for $u - t \leq 1$ gives

$$p(t, x; u, y) \geq \frac{1}{C(u - t)^{N/2}}\exp\left[-C(u - t) - \frac{C\,|y - x|^2}{u - t}\right]$$

for arbitrary $u-t>0$. Finally, to remove the restriction that $\Lambda \leqslant 1$, we repeat the scaling argument given in Lemma 3.3.

In the statement of the lower bound that follows there appears a probability density function ρ, which may be chosen subject to certain restrictions. A probability density function ρ will be called *admissible* if the following quantities, the *characteristics* of ρ, are finite:

$$B(\rho) \equiv \sup_{z\in\mathbb{R}^N} \rho(z), \quad H(\rho) \equiv \int_{\{\rho>0\}} \frac{|\nabla\rho|^2}{\rho}\,dz, \quad S(\rho) \equiv \int_{\mathbb{R}^N} |z|^2 \rho(z)\,dz.$$

Any Gaussian is admissible. There are also admissible functions of compact support. Notice however that any bound on H prevents ρ from being too close to a delta-function. Given a probability density function $\rho=\rho_1$, we define for $0<\tau<\infty$,

$$\rho_\tau(z) = \tau^{-N/2}\rho_1(\tau^{-\frac{1}{2}}z), \quad \text{where } z \in \mathbb{R}^N.$$

Notice that $B(\rho_\tau)=\tau^{-N/2}B(\rho_1)$, $H(\rho_\tau)=H(\rho_1)/\tau$ and $S(\rho_\tau)=S(\rho_1)\tau$.

We define the *smoothed energy function* E_β: for $0<\beta<\infty$, $t<u$ and $x, y \in \mathbb{R}^N$, set

$$E_\beta(t,x\,;u,y) = \inf\left\{\frac{1}{4}\int_t^u \rho_{\beta(u-t)} * (|\dot\gamma_s - a(b-\hat{b})|^2_{a^{-1}})(s,\gamma_s)\,ds: \right.$$
$$\left. \gamma \in C^1([t,u],\mathbb{R}^N) \text{ with } \gamma_t = x,\ \gamma_u = y\right\}.$$

We should more precisely call E_β 'an energy function associated with the smoothings of certain functions associated with the coefficients': it is not a smoothing of E. Notice that the crude bounds (2.3) apply to E_β as well as E. In particular, for $C=C(\lambda,N)$, the expression $C[\Lambda(u-t)+E_\beta(t,x\,;u,y)]$ is equivalent to $C[\Lambda(u-t)+E(t,x\,;u,y)]$.

THEOREM 3.5 (the lower bound). *Let ρ be admissible with characteristics B, H and S. For all $\alpha \in (\frac{1}{2},1)$ satisfying $\alpha^2/(2\alpha-1)>\lambda^2$, there is a constant $C(\alpha,\lambda,B,H,N,S)<\infty$ such that, for all $t, u \in \mathbb{R}$ with $t<u$ and all $x,y\in\mathbb{R}^N$, for $\beta=(1+\Lambda(u-t)+E(t,x\,;u,y))^{-1/(4\alpha-1)}$,*

$$p(t,x\,;u,y) \geqslant \frac{1}{(u-t)^{N/2}} \exp\left\{-E_\beta(t,x\,;u,y) + \int_t^u \inf_{z\in\mathbb{R}^N} c(s,z)\,ds \right.$$
$$- C[(1+\Lambda(u-t)+E_\beta(t,x\,;u,y))^{1/(4\alpha-1)}$$
$$\left. + (1+\Lambda(u-t)+E_\beta(t,x\,;u,y))^{1/2(4\alpha-1)}(\Lambda(u-t))^{\frac{1}{2}}]\right\}.$$

Proof. The degree of smoothing in the smoothed energy function E_β is proportional to $u-t$. In the proof we consider a smoothed energy function in which the degree of smoothing is fixed: define

$$E^\tau(t,x\,;u,y) = \inf\left\{\frac{1}{4}\int_t^u \rho_\tau * (|\dot\gamma_s - a(b-\hat{b})|^2_{a^{-1}})(s,\gamma_s)\,ds: \right.$$
$$\left. \gamma \in C^1([t,u],\mathbb{R}^N) \text{ with } \gamma_t = x,\ \gamma_u = y\right\}.$$

Fix $\varepsilon \in (0, \frac{1}{4}]$ and set $t' = t + \varepsilon(u-t)$, $u' = u - \varepsilon(u-t)$. By Theorem A.6 there is a constant $C(\alpha, \lambda) < \infty$ such that

$$(3.4)\quad E^{\tau}(t', x\,; u', y) \leqslant E^{\tau}(t, x\,; u, y) + C[E^{\tau}(t, x\,; u, y) + \Lambda(u-t)]\varepsilon^{2\alpha-1}.$$

(*Warning*: different notation is used in the Appendix.) From Theorem 3.4 we deduce that for all $\Lambda \in [0, \infty)$, $\delta \in (0, \infty)$ and $t < u$,

$$(3.5)\quad \int_{\mathbb{R}^N} \rho_{\delta(u-t)}(x) \log p(t, x\,; u, 0)\, dx \geqslant -\tfrac{1}{2}N \log(u-t) - C(\lambda, N)(1 + \delta S + \Lambda(u-t)).$$

Notice the improvement over (3.3). We will use this estimate, together with essentially the same argument as used for Theorem 3.4, to show that there is a constant $C(\lambda, B, H, N, S) < \infty$ such that for all $\beta \in (0, \infty)$, $t < u$ and $x, y \in \mathbb{R}^N$,

$$(3.6)\quad p(t, x\,; u, y) \geqslant \frac{1}{(u-t)^{N/2}} \exp\Big\{ -E^{\beta(u-t)}(t', x\,; u', y) + \int_t^u \inf_{z \in \mathbb{R}^N} c(s, z)\, ds - C\Big[1 + \frac{\beta}{\varepsilon} + \frac{1}{\beta} + \varepsilon\Lambda(u-t) + \Big(\frac{\Lambda(u-t)}{\beta}\Big)^{\frac{1}{2}}\Big]\Big\}.$$

Noting that $E^{\beta(u-t)}(t, x\,; u, y) = E_\beta(t, x\,; u, y)$, we see that the theorem follows on taking

$$\beta = (1 + \Lambda(u-t) + E(t, x\,; u, y))^{-1/(4\alpha-1)},$$
$$\varepsilon = (8 + \Lambda(u-t) + E(t, x\,; u, y))^{-2/(4\alpha-1)}$$

and using (3.4).

Take any $\gamma \in C^1([t', u'], \mathbb{R}^N)$ with $\gamma_{t'} = x$ and $\gamma_{u'} = y$ and extend γ to $[t, u]$ by

$$\gamma(s) = \begin{cases} x & \text{if } t \leqslant s \leqslant t', \\ y & \text{if } u' \leqslant s \leqslant u. \end{cases}$$

Consider the new heat kernel

$$p^{\gamma}(s, z\,; s', z') = p(s, z + \gamma(s)\,; s', z' + \gamma(s')).$$

The coefficients associated with p^{γ} are listed in § 1, (4). We have for $\delta \in (0, \infty)$, by (3.5),

$$\int_{\mathbb{R}^N} \rho_{\delta(t'-t)}(z) \log p^{\gamma}(t, 0\,; t', z)\, dz = \int_{\mathbb{R}^N} \rho_{\delta(t'-t)}(z) \log p(t, x\,; t', x+z)\, dz \geqslant -\tfrac{1}{2}N \log \varepsilon(u-t) - C(\lambda, N)(1 + \delta S + \Lambda\varepsilon(u-t)).$$

Set $v = \frac{1}{2}(t+u)$. Let ϕ tend to a delta-function in (3.2) to obtain for $s \in [t', v]$,

$$\frac{d}{ds}\int_{\mathbb{R}^N} \rho_{\delta\varepsilon(u-t)}(z) \log p^{\gamma}(t, 0\,; s, z)\, dz \geqslant -\frac{C(\lambda, H, N)}{\delta\varepsilon(u-t)}\Big(1 + \frac{1}{\varepsilon'}\Big) - \varepsilon'\Lambda + \inf_{z \in \mathbb{R}^N} c(s, z) - \tfrac{1}{4}(\rho_{\delta\varepsilon(u-t)} * |\dot{\gamma}_s - a(b - \hat{b})|^2_{a^{-1}})(s, \gamma_s).$$

Integrating this inequality from $s=t'$ to $s=v$ and adding the result to the one above we get

$$\int_{\mathbb{R}^N} \rho_{\delta\varepsilon(u-t)}(z)p^{\gamma}(t,0;v,z)\,dz$$
$$\geqslant -\tfrac{1}{2}N\log\varepsilon(u-t) - C\Big[1+\delta+(\varepsilon+\varepsilon')\Lambda(u-t)+\frac{1}{\delta\varepsilon}\Big(1+\frac{1}{\varepsilon'}\Big)\Big]$$
$$+\int_{t'}^{v}\inf_{z\in\mathbb{R}^N} c(s,z)\,ds - \frac{1}{4}\int_{t'}^{v}(\rho_{\delta\varepsilon(u-t)} * |\dot\gamma_s - a(b-\hat b)|^2_{a^{-1}})(s,\gamma_s)\,ds.$$

By time reversal there is a corresponding lower bound for

$$\int_{\mathbb{R}^N}\rho_{\delta\varepsilon(u-t)}(z)p^{\gamma}(v,z;u,0)\,dz.$$

Just as in the proof of Theorem 3.4, by Jensen's inequality,

$$\log p(t,x;u,y) = \log p^{\gamma}(t,0;u,0)$$
$$\geqslant -\log B + \tfrac{1}{2}N\log\delta\varepsilon(u-t) + \int_{\mathbb{R}^N}\rho_{\delta\varepsilon(u-t)}(z)p^{\gamma}(t,0;v,z)\,dz$$
$$+\int_{\mathbb{R}^N}\rho_{\delta\varepsilon(u-t)}(z)p^{\gamma}(v,z;u,0)\,dz$$
$$\geqslant -\tfrac{1}{2}N\log(u-t) - C\Big[1+\delta+(\varepsilon+\varepsilon')\Lambda(u-t)+\frac{1}{\delta\varepsilon}\Big(1+\frac{1}{\varepsilon'}\Big)\Big]$$
$$+\int_{t}^{u}\inf_{z\in\mathbb{R}^N} c(s,z)\,ds - \frac{1}{4}\int_{t'}^{u'}(\rho_{\delta\varepsilon(u-t)} * |\dot\gamma_s - a(b-\hat b)|^2_{a^{-1}})(s,\gamma_s)\,ds.$$

Taking $\delta=\beta/\varepsilon$ and maximizing over γ and ε' we arrive at (3.6).

We now consider the case of Hölder continuous coefficients.

LEMMA 3.6. *Let a and $(b-\hat b)/\Lambda^{\frac{1}{2}}$ be uniformly Hölder continuous in space of exponent $\alpha\in(0,1]$ and constant $A\in(0,\infty)$. Suppose that* $\operatorname{supp}\rho_1\subseteq\{|z|\leqslant 1\}$. *Then there is a constant $C(\lambda)<\infty$ such that, for all $t<u$ and $x, y\in\mathbb{R}^N$,*

$$E_\beta(t,x;u,y)\leqslant E(t,x;u,y)+C(E(t,x;u,y)+\Lambda(u-t))A\beta^{\alpha/2}(u-t)^{\alpha/2}.$$

Proof. Set $\tau=\beta(u-t)$. Note that $\operatorname{supp}\rho_\tau\subseteq\{|z|\leqslant\tau^{\frac{1}{2}}\}$. Consider any path γ from (t,x) to (u,y). We have, in an obvious notation,

$$I^{\tau}(\gamma) = I(\gamma)+\int_t^u(|\dot\gamma_s - a(b-\hat b)|^2_a) * (\rho_0-\rho_\tau)(s,\gamma_s)\,ds$$
$$\leqslant I(\gamma)+C(\lambda)(I(\gamma)+\Lambda(u-t))A\tau^{\alpha/2}.$$

Now minimise over γ.

THEOREM 3.7 (the lower bound for Hölder continuous coefficients). *Let a and $(b-\hat b)/\Lambda^{\frac{1}{2}}$ be Hölder continuous in time of exponent $\alpha\in(0,1]$ and constant $A\in[0,\infty)$, and let $|b-\hat b|^2_a/\Lambda^2$ be Hölder continuous in space of exponent α and constant $A/\Lambda^{\alpha/2}$.*

(a) *For any admissible probability density function ρ_1 with characteristics B, H and S, there is a constant $C(\alpha, \lambda, B, H, N, S) < \infty$ such that, for all $t < u$ and $x, y \in \mathbb{R}^N$, setting*

$$\beta = [(1 + A^{1/\alpha}(u-t))(1 + \Lambda(u-t) + E(t, x; u, y))]^{-\frac{1}{3}},$$

we have

$$\begin{aligned} p(t, x; u, y) \geq \frac{1}{(u-t)^{N/2}} \exp\Big\{ & -E_\beta(t, x; u, y) + \int_t^u \inf_{z \in \mathbb{R}^N} c(s, z)\, ds \\ & - C[(1 + A^{1/\alpha}(u-t))^{\frac{1}{3}}(1 + \Lambda(u-t) + E_\beta(t, x; u, y))^{\frac{1}{3}} \\ & + (\Lambda(u-t))^{\frac{1}{2}}(1 + A^{1/\alpha}(u-t))^{\frac{1}{6}}(1 + \Lambda(u-t) + E_\beta(t, x; u, y))^{\frac{1}{6}}]\Big\}. \end{aligned}$$

(b) *Suppose further that a and $(b - \hat{b})/\Lambda^{\frac{1}{2}}$ are Hölder continuous in space of exponent α and constant $A^{\frac{1}{2}}$. Then there is a constant $C(\alpha, \lambda, N) < \infty$ such that for all $t, u \in \mathbb{R}$ with $t < u$ and all $x, y \in \mathbb{R}^N$,*

$$\begin{aligned} p(t, x; u, y) \geq \frac{1}{(u-t)^{N/2}} \exp\Big\{ & -E(t, x; u, y) + \int_t^u \inf_{z \in \mathbb{R}^N} c(s, z)\, ds \\ & - C[\mu^{2/(\alpha+2)} \nu^{\alpha/(\alpha+2)} + \mu^{(\alpha+1)/(\alpha+2)} \nu^{1/(\alpha+2)} \\ & + (\Lambda(u-t))^{\frac{1}{2}} (\mu^{2/(\alpha+2)} \nu^{\alpha/(\alpha+2)})^{\frac{1}{2}}]\Big\}, \end{aligned}$$

where $\mu = 1 + \Lambda(u-t) + E(t, x; u, y)$ and $\nu = 1 + A^{1/\alpha}(u-t)$.

Proof. (a) The proof is identical to Theorem 3.5 except that, instead of Theorem A.6, we can now use Theorem A.7 to find a constant $C(\alpha, \lambda) < \infty$, such that

$$\begin{aligned} (3.7) \quad E^\tau(t', x; u', y) \leq\ & E^\tau(t, x; u, y) \\ & + C[(1 + A^{1/\alpha}(u-t))(\Lambda(u-t) + E^\tau(t, x; u, y))]\varepsilon. \end{aligned}$$

Set $\beta = [(1 + A^{1/\alpha}(u-t))(1 + \Lambda(u-t) + E(t, x; u, y))]^{-\frac{1}{3}}$, and $\varepsilon = \frac{1}{4}\beta^2$ in (3.6), then, using (3.7) in place of (3.5), we obtain the result.

(b) Choose ρ_1 in (a) to have support inside the unit ball. Then, by Lemma 3.6, there is a constant $C(\lambda) < \infty$ such that

$$E_\beta(t, x; u, y) \leq E(t, x; u, y) + C(E(t, x; u, y) + \Lambda(u-t)) A^{\frac{1}{2}} (u-t)^{\alpha/2} \beta^{\alpha/2}.$$

Hence by (3.6) and (3.7),

$$\begin{aligned} p(t, x; u, y) \geq \frac{1}{(u-t)^{N/2}} \exp\Big\{ & -E(t, x; u, y) + \int_t^u \inf_{z \in \mathbb{R}^N} c(s, z)\, ds \\ & - C\Big[1 + \frac{\beta}{\varepsilon} + \frac{1}{\beta} + \mu\nu\varepsilon + \mu(\nu\beta)^{\alpha/2} + \Big(\frac{\Lambda(u-t)}{\beta}\Big)^{\frac{1}{2}}\Big]\Big\}. \end{aligned}$$

The result follows on optimising over ε and β.

Appendix. Continuity of the energy function

Write $\Gamma(t, x ; u, y)$ for the set of absolutely continuous paths from x at time t to y at time u; that is,

$$\Gamma(t, x ; u, y)=\left\{\gamma \in C([t, u], \mathbb{R}^N):\ \gamma_t=x,\ \gamma_u=y \text{ and } \int_t^u |\dot{\gamma}_s|^2\, ds<\infty\right\}.$$

Let a be a continuous function on $\mathbb{R} \times \mathbb{R}^N$ with values in the set of $N \times N$ real symmetric matrices. Let b and c be continuous functions on $\mathbb{R} \times \mathbb{R}^N$ with values in $\mathbb{R}^N$ and $\mathbb{R}$ respectively. We suppose for certain constants $\lambda \in [1, \infty)$ and $\Delta \in (0, \infty)$ that, uniformly on $\mathbb{R} \times \mathbb{R}^N$,

(A.1) $$\lambda^{-1} I \leqslant a \leqslant \lambda I, \quad |b| \leqslant \lambda \Delta, \quad |c| \leqslant \Delta^2.$$

The *energy* of a path $\gamma \in \Gamma(t, x ; u, y)$ (for given coefficients a, b and c) is defined by

$$I(\gamma)=\int_t^u [\langle \dot{\gamma}_s, a(s, \gamma_s)\dot{\gamma}_s\rangle + \langle \dot{\gamma}_s, b(s, \gamma_s)\rangle + c(s, \gamma_s)]\, ds,$$

and the *energy function* by

(A.2) $$E(t, x ; u, y)=\inf_{\gamma \in \Gamma(t,x;u,y)} I(\gamma).$$

In earlier sections we defined the energy of a path $\gamma \in \Gamma(t, x ; u, y)$ to be

$$\frac{1}{4}\int_t^u |\dot{\gamma}_s - a(b-\hat{b})|^2_{a^{-1}}(s, \gamma_s)\, ds,$$

where a, b and $\hat{b}$ satisfied the bounds

$$\lambda^{-1} I \leqslant a \leqslant \lambda I, \quad |b|_a^2 + |\hat{b}|_a^2 \leqslant \Lambda.$$

If we take $\Delta = \Lambda^{\frac{1}{2}}$ and make a and $b-\hat{b}$ continuous, this notion of energy falls into the set-up we are now considering. Moreover, if we take $\Delta = \Lambda^{\frac{1}{2}}$ (no continuity required) the smoothed energy function of § 3 falls into our set-up. *Of course the quantities a, b and c do not play the same rôles as in the earlier sections.*

We show in Theorem A.6 that the energy function E is Hölder continuous with an exponent depending on λ. We also give an example to show that for λ sufficiently large E may fail to be Hölder continuous of any given exponent. We show further in Theorem A.7 that if the coefficients a, b and c are themselves Hölder continuous, then E is Lipschitz continuous.

We begin by noting some easy facts. There is a constant $C(\lambda) \in (0, \infty)$ such that

(A.3) $$\int_t^u |\dot{\gamma}_s|^2\, ds \leqslant C[I(\gamma) + \Delta^2(u-t)]$$

and

(A.4) $$I(\gamma) \leqslant C\left[\int_t^u |\dot{\gamma}_s|^2\, ds + \Delta^2(u-t)\right].$$

It follows that

(A.5) $$\frac{|y-x|^2}{u-t} \leqslant C[E(t, x ; u, y) + \Delta^2(u-t)]$$

and

$$E(t, x\,; u, y) \leq C\left[\frac{|y-x|^2}{u-t} + \Delta^2(u-t)\right]. \tag{A.6}$$

Since I is lower semi-continuous on $\Gamma(t, x\,; u, y)$ for the topology of weak convergence, (A.3) and the usual sort of compactness argument show that the infimum in (A.2) is attained: minimal energy paths γ^* always exist. Having established this fact, we will not need the hypothesis of continuity of a, b and c again. In particular, in cases of discontinuous coefficients where we know for other reasons that minimal energy paths exist, the results of the appendix still work. For paths of minimal energy we have, for $t \leq s < s' \leq u$,

$$\begin{aligned} |\gamma^*_{s'} - \gamma^*_s| &\leq \int_s^{s'} |\dot\gamma^*_r|\, dr \leq (s'-s)^{\frac{1}{2}}\left(\int_s^{s'} |\dot\gamma^*_r|^2\, dr\right)^{\frac{1}{2}} \\ &\leq C[|y-x| + \Delta(u-t)]\left(\frac{s'-s}{u-t}\right)^{\frac{1}{2}}. \end{aligned} \tag{A.7}$$

Taking $s = t$ and $s' = u$, we find that this provides a bound on the length of any minimal energy path. It also shows such paths must be Hölder continuous of exponent $\frac{1}{2}$; it is a key step in showing the energy function is continuous to show that this exponent can be improved to some $\beta > \frac{1}{2}$.

The following example is intended to show that the energy function may fail to be Lipschitz continuous. The coefficient a which is chosen is only lower semi-continuous, but this is still enough to make I lower semi-continuous. In any case we exhibit minimal energy paths.

Example. Let the dimension $N = 1$. Given $\beta \in (\frac{1}{2}, 1)$, choose $\lambda > \beta/(2\beta - 1)^{\frac{1}{2}}$. Take

$$a(s, z) = \begin{cases} \lambda & \text{if } s > 0 \text{ and } z < s^\beta, \\ \lambda^{-1} & \text{otherwise.} \end{cases}$$

Take $b = 0$ and $c = 0$. Then

(1) *the unique path in* $\Gamma(0, 0\,; 1, 1)$ *of minimal energy is given by* $s \mapsto s^\beta$,

(2) *there is a constant* $C(\beta, \lambda) \in (0, \infty)$ *with the property that*

$$E(0, 0\,; 1, 1) - E(0, x\,; 1, 1) = Cx^{2-1/\beta} \quad \textit{for all } x \in [0, 1-\beta].$$

To see (1), first note that any minimal energy path in $\Gamma(0, 0\,; 1, 1)$ will consist of arcs of the curve $s \mapsto s^\beta$ and/or chords joining points on the curve; it is an easy exercise to show that every arc has a lower energy than the chord between its endpoints, which implies (i). For $x \in [0, 1-\beta]$, the unique path in $\Gamma(0, x\,; 1, 1)$ of minimal energy has the form

$$\gamma^*(s) = \begin{cases} x + \eta^* s & \text{if } s \in [0, s^*], \\ s^\beta & \text{if } s \in (s^*, 1], \end{cases}$$

where η^* and s^* are determined by the requirement that γ^* and $\dot\gamma^*$ be continuous at s^*. It is now simple to calculate $C(\beta, \lambda)$ and establish (2).

In dimension 1, for $x \leq y$, we write $\Gamma_+(t, x\,; u, y)$ for the set of non-decreasing elements of $\Gamma(t, x\,; u, y)$. This set also contains an element of minimal energy.

LEMMA A.1. *Let the dimension $N=1$. Let $\beta \in (\frac{1}{2}, 1)$ satisfying $\beta^2/(2\beta-1) > \lambda^2$ be given. Let $\Delta = 1$. Then there is a constant $C(\beta, \lambda) \in (0, \infty)$ such that, for all $y \geq 0$, for all $\gamma^* \in \Gamma_+(0, 0; 1, y)$ of minimal energy, for all $s \in [0, 1]$,*

$$\gamma_s^* \leq C[y+1]s^{\beta}. \tag{A.8}$$

Proof. Fix $\alpha = \alpha(\beta, \lambda) \in (\beta, 1)$ for which $\alpha^2/(2\alpha-1) > \lambda^2$ and set

$$\delta = \left(\frac{\alpha}{(2\alpha-1)^{\frac{1}{2}}} - \lambda\right)^{\frac{1}{2}};$$

define $\varepsilon = \varepsilon(\beta, \lambda) \in (0, 1)$ by the equation

$$1 - \varepsilon^{(\alpha/\beta - 1)(2\alpha-1)} = (2\alpha-1)^{\frac{1}{2}}\lambda/\alpha,$$

and set

$$C = \min\left\{\frac{1}{\varepsilon^{\alpha}}, \frac{4\lambda}{\delta^2} + \frac{2\lambda^{\frac{1}{2}}}{\delta}\right\}.$$

Suppose that $\gamma \in \Gamma_+(0, 0; 1, y)$ and that $\gamma_{s_0} \geq Ks_0^{\beta}$ for some $s_0 \in (0, 1]$, where $K = C(1+y)$. Then

$$\gamma_s \geq \gamma_{s_0} \geq Ks^{\alpha} \quad \text{for all } s_0 \leq s \leq s_0^{\beta/\alpha}.$$

Thus, if

$$\xi = \min\{s \in [0, s_0]:\ \gamma_t \geq Kt^{\alpha} \text{ for } t \in [s, s_0]\}$$

and

$$\eta = \max\{s \in [s_0, 1]:\ \gamma_t \geq Kt^{\alpha} \text{ for } t \in [s_0, s]\},$$

then $\xi \leq s_0$ and $\eta \geq s_0^{\beta/\alpha}$, and therefore $\xi \leq \eta^{\alpha/\beta}$. Moreover, because $K\varepsilon^{\alpha} \geq y \geq \gamma_\eta \geq K\eta^{\alpha}$, we know that $\eta \leq \varepsilon$. In particular, this means that $\eta < 1$ and therefore that $\gamma_\eta = K\eta^{\alpha}$. Thus, if we define

$$\gamma'_s = \begin{cases} (s/\eta)\gamma_\eta & \text{for } s \in [0, \eta], \\ \gamma_s & \text{for } s \in [\eta, 1], \end{cases}$$

then $\gamma' \in \Gamma_+(0, 0; 1, y)$ and

$$I(\gamma) - I(\gamma') = I(\gamma\,|_{[0,\eta]}) - I(\gamma'\,|_{[0,\eta]}) \geq \frac{1}{\lambda}\int_0^{\eta} \dot{\gamma}_s^2\, ds - \lambda\int_0^{\eta} \dot{\gamma}'^2\, ds - 2\lambda(\gamma_\eta + \eta).$$

The key to our argument lies in the observation that, for $g_s = Ks^{\alpha}$,

$$\int_0^{\eta} \dot{\gamma}_s^2\, ds \geq \int_{\xi}^{\eta} \dot{\gamma}_s^2\, ds \geq \int_{\xi}^{\eta} \dot{g}_s^2\, ds. \tag{A.9}$$

To see the second of these, set $\phi = \gamma - g$ on $[\xi, \eta]$. Then $\phi \geq 0$ and $\phi(\xi) = \phi(\eta) = 0$. Hence, after integrating by parts and using $\ddot{g} \leq 0$, one sees that

$$\int_t^{\eta} \dot{\gamma}_s^2\, ds \geq \int_t^{\eta} \dot{g}_s^2\, ds - \phi_t \dot{g}_t,$$

for any $t \in (\xi, \eta]$. When $\xi > 0$, the preceding immediately extends to $t = \xi$ and yields the desired result. When $\xi = 0$, one gets the same conclusion, only here it is necessary to recall that $\alpha > \frac{1}{2}$ and that ϕ is Hölder continuous of order $\frac{1}{2}$. Plugging (A.9) into our estimate for $I(\gamma) - I(\gamma')$ and recalling that $\xi \leq \eta^{\alpha/\beta}$, we find that

$$I(\gamma)-I(\gamma')\geq \eta^{2\alpha-1}[(K\delta)^2-2\lambda(K\eta^{1-\alpha}+\eta^{2(1-\alpha)})]>0.$$

Thus γ is not of minimal energy, and so any path γ^* of minimal energy must satisfy (A.8).

LEMMA A.2. *Let the dimension $N=1$. Suppose that $a(0,z)=1$ for all $z\in\mathbb{R}$, and that $\Delta=1$. Suppose also that a and b are Hölder continuous in time and c is Hölder continuous in space, all of exponent $\alpha\in(0,1]$ and constant $A\in[0,\infty)$. Then there is a constant $C(\alpha,\lambda)\in(0,\infty)$ such that, for all $y\geq 0$, all $\gamma^*\in\Gamma_+(0,0;1,y)$ of minimal energy, and all $s\in[0,1]$,*

$$\gamma_s^*\leq C[(y+1)(1+A^{1/\alpha})^{\frac{1}{2}}]s. \tag{A.10}$$

Proof. We will show that, for all $s\in[0,1]$,

$$\frac{\gamma_{s/2}^*}{s/2}\leq \max\left\{1,(1+3A^{\frac{1}{2}}s^{\alpha/2})\frac{\gamma_s^*}{s}\right\}. \tag{A.11}$$

Bearing in mind that γ^* is non-decreasing, we may restrict attention to the case where

$$\gamma_{s/2}^*=\tfrac{1}{2}(1+\varepsilon)\gamma_s^*\quad\text{for some }\varepsilon\in[0,1] \tag{A.12}$$

and

$$\gamma_s^*\geq\tfrac{1}{2}s. \tag{A.13}$$

Then by (A.12) we have

$$\int_0^s \dot\gamma_r^{*2}\,dr\geq(1+\varepsilon^2)\frac{\gamma_s^{*2}}{s}. \tag{A.14}$$

Define $\gamma\in\Gamma_+(0,0;1,y)$ by

$$\gamma_r=\begin{cases}(r/s)\gamma_s^* & \text{for } r\in[0,s],\\ \gamma_r^* & \text{for } r\in[s,1];\end{cases}$$

then

$$\int_0^s \dot\gamma_r^2\,dr=\frac{\gamma_s^{*2}}{s}. \tag{A.15}$$

By minimality

$$\begin{aligned}0\leq I(\gamma)-I(\gamma^*)&=I(\gamma\,|_{[0,s]})-I(\gamma^*\,|_{[0,s]})\\ &\leq(1+As^\alpha)\int_0^s\dot\gamma_r^2\,dr-(1-As^\alpha)\int_0^s\dot\gamma_r^{*2}\,dr+A(\gamma_s^*s^\alpha+s\gamma_s^{*\alpha}),\end{aligned}$$

so, by (A.13), (A.14) and (A.15) we find that $\varepsilon\leq 3A^{\frac{1}{2}}s^{\alpha/2}$, establishing (A.11).

Observe that

$$\prod_{n=0}^{\infty}(1+3(2^{-n})^{\alpha/2})<\infty\quad\text{for all }\alpha\in(0,1].$$

Take $t=(1+A^{1/\alpha})^{-1}$; then by (A.7),

$$\gamma_t^*/t\leq C(\lambda)[(y+1)(1+A^{1/\alpha})^{\frac{1}{2}}].$$

But by (A.11), for $s=2^{-n}t$, $n=0, 1, 2, \ldots$,

$$\gamma_s^*/s \leqslant \prod_{k=0}^{n-1} (1+3(2^{-n})^{\alpha/2})(1+\gamma_t^*/t) \leqslant C(\alpha, \lambda)[(y+1)(1+A^{1/\alpha})^{\frac{1}{2}}],$$

for some $C(\alpha, \lambda) \in (0, \infty)$. Since γ^* is non-decreasing, such a bound therefore holds for all $s \in [0, 1]$.

LEMMA A.3. *Let $\Delta = 1$ and let $\beta \in (\frac{1}{2}, 1)$ satisfying $\beta^2/(2\beta - 1) > \lambda^2$ be given. Then there is a constant $C(\beta, \lambda) \in (0, \infty)$ such that, for all $y \in \mathbb{R}^N$, for all $\gamma^* \in \Gamma(0, 0; 1, y)$, for all $s \in [0, 1]$,*

$$|\gamma_s^*| \leqslant C[|y|+1]s^{\beta}.$$

Proof. By (A.7) we have

$$\sigma \equiv \int_0^1 |\dot{\gamma}_s^*| \, ds \leqslant C(\lambda)[|y|+1].$$

We can write $\gamma^* = \gamma \circ g^*$ where $\gamma \in \Gamma(0, 0; \sigma, y)$ is parametrised by arc-length and $g^* \in \Gamma_+(0, 0; 1, \sigma)$. Define for $s \in [0, 1]$ and $\xi \in [0, \sigma]$,

$$\tilde{a}(s, \xi) = \langle \dot{\gamma}_\xi, a(s, \gamma_\xi)\dot{\gamma}_\xi \rangle,$$
$$\tilde{b}(s, \xi) = \langle \dot{\gamma}_\xi, b(s, \gamma_\xi) \rangle,$$
$$\tilde{c}(s, \xi) = c(s, \gamma_\xi).$$

Write $\tilde{I}$ for the energy on $\Gamma_+(0, 0; 1, \sigma)$ associated with coefficients $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$. Then, in an obvious notation, $\tilde{\lambda} = \lambda$, $\tilde{\Delta} = \Delta = 1$ and $I(\gamma^*) = I(\gamma \circ g^*) = \tilde{I}(g^*)$. Since γ^* minimises I, g^* must minimise $\tilde{I}$, so by Lemma A.1, for some $C(\beta, \lambda) \in (0, \infty)$,

$$|\gamma_s^*| = |\gamma_{g^*(s)}| \leqslant g^*(s) \leqslant C[\sigma + 1]s^{\beta}.$$

LEMMA A.4. *Let $\Delta = 1$. Suppose that a and b are Hölder continuous in time and c is Hölder continuous in space, all of exponent $\alpha \in (0, 1]$ and constant $A \in [0, \infty)$. Then there is a constant $C(\alpha, \lambda) \in (0, \infty)$ such that, for all $y \in \mathbb{R}^N$, all $\gamma^* \in \Gamma(0, 0; 1, y)$, and all $s \in [0, 1]$,*

$$|\gamma_s^*| \leqslant C[(|y|+1)(1+A^{1/\alpha})^{\frac{1}{2}}]s.$$

Proof. By (A.7) we have

$$\sigma \equiv \int_0^1 |\dot{\gamma}_s^*|_{a(0, \gamma_s^*)} \, ds \leqslant C(\lambda)[|y|+1].$$

We can write $\gamma^* = \gamma \circ g^*$ where $\gamma \in \Gamma(0, 0; \sigma, y)$ is parametrised by the arc-length associated with the Riemannian metric $a(0, \cdot)$, and $g^* \in \Gamma_+(0, 0; 1, \sigma)$. Define $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ and $\tilde{I}$ as in the proof of Lemma A.3. Then, in an obvious notation, $\tilde{a}(0, \xi) = 1$ for all $\xi \in [0, \sigma]$, $\tilde{\lambda} = \lambda^2$, $\tilde{\Delta} = \Delta = 1$, $\tilde{\alpha} = \alpha$ and $\tilde{A} = \lambda A$. We have $I(\gamma^*) = I(\gamma \circ g^*) = \tilde{I}(g^*)$ and since γ^* minimises I, g^* must minimise $\tilde{I}$. Hence, by Lemma A.2, for some $C(\alpha, \lambda) \in (0, \infty)$,

$$|\gamma_s^*| = |\gamma_{g^*(s)}| \leqslant \lambda g^*(s) \leqslant C[(\sigma+1)(1+A^{1/\alpha})^{\frac{1}{2}}]s.$$

LEMMA A.5. *Let $\Delta = 1$. Suppose for some $\beta \in (\frac{1}{2}, 1]$ and $K \in [1, \infty)$, for some $\gamma^* \in \Gamma(0, 0; 1, y)$ of minimal energy, we have*

$$|\gamma_s^*| \leq K s^\beta \quad \text{for all } s \in [0, 1].$$

There is a constant $C(\lambda)$ such that for all $s \in [-1, \frac{1}{2}]$ and all $x \in \mathbb{R}^N$,

$$E(s, 0; 1, y) - E(0, 0; 1, y) \leq CK^2 s^{2\beta-1} \tag{A.15}$$

and

$$E(0, x; 1, y) - E(0, 0; 1, y) \leq CK^{1/\beta} |x|^{2-1/\beta}. \tag{A.16}$$

Proof. First consider the case where $s \in [-1, 0]$. Define $\gamma \in \Gamma(s, 0; 1, y)$ by

$$\gamma|_{[s,0]} \equiv 0, \quad \gamma|_{[0,1]} = \gamma^*.$$

Then

$$E(s, 0; 1, y) - E(0, 0; 1, y) \leq I(\gamma) - I(\gamma^*) = \int_s^0 c(r, 0)\, dr \leq s.$$

Now consider the case where $s \in (0, \frac{1}{2}]$. Choose $\gamma \in \Gamma(s, 0; 1, y)$ such that

$$I(\gamma|_{[s,2s]}) = E(s, 0; 2s, \gamma_{2s}^*) \quad \text{and} \quad \gamma|_{[2s,1]} = \gamma^*|_{[2s,1]}.$$

Then, by (A.6),

$$\begin{aligned} E(s, 0; 1, y) - E(0, 0; 1, y) &\leq I(\gamma) - I(\gamma^*) \leq E(s, 0; 2s, \gamma_{2s}^*) \\ &\leq C(\lambda)[|\gamma_{2s}^*|^2/s + s] \leq CK^2 s^{2\beta-1}. \end{aligned}$$

This completes the proof of (A.15); we turn to (A.16). Set $s = (|x|/K)^{1/\beta}$ and choose $\gamma \in \Gamma(0, x; 1, y)$ such that

$$I(\gamma|_{[0,s]}) = E(0, x; s, \gamma_s^*) \quad \text{and} \quad \gamma|_{[s,1]} = \gamma^*|_{[s,1]}.$$

Then, by (A.6),

$$\begin{aligned} E(0, x; 1, y) - E(0, 0; 1, y) &\leq I(\gamma) - I(\gamma^*) \leq E(0, x; s, \gamma_s^*) \\ &\leq C(\lambda)[|\gamma_s^* - x|^2/s + s] \leq CK^{1/\beta} |x|^{2-1/\beta}. \end{aligned}$$

We will use a scaling argument below. For $\rho, \sigma \in (0, \infty)$, set $z' = \rho z$ and $s' = \sigma s$; set $a'(s', z') = a(s, z)$, $b'(s', z') = (\rho/\sigma) b(s, z)$, $c'(s', z') = (\rho/\sigma)^2 c(s, z)$ and $\gamma'_{s'} = (\gamma_s)'$. Then in an obvious notation, $\lambda' = \lambda$, $\Delta' = (\rho/\sigma)\Delta$, $I'(\gamma') = (\rho^2/\sigma) I(\gamma)$ and $E'(t', x'; u', y') = (\rho^2/\sigma) E(t, x; u, y)$. We will need to impose a condition of Hölder continuity on the coefficients for one of our results. We do this in a way which transforms well under the scaling: the requirement that *a and b/Δ be Hölder continuous in time of exponent $\alpha \in (0, 1]$ and constant $A \in [0, \infty)$, and c/Δ^2 be Hölder continuous in space of exponent α and constant A/Δ^α*, is equivalent to the corresponding requirement in which a, b, c and Δ are replaced by a', b', c' and Δ' and A by $A' = A/\sigma^\alpha$.

THEOREM A.6. *Let $\beta \in (\frac{1}{2}, 1)$ satisfying $\beta^2/(2\beta - 1) > \lambda^2$ be given. Then there is a constant $C(\beta, \lambda) \in (0, \infty)$ such that for all $t, u \in \mathbb{R}$ with $t < u$, and all $x, y \in \mathbb{R}^N$,*

$$\begin{aligned} &|E(t', x; u, y) - E(t, x; u, y)| \\ &\quad \leq C\left[\frac{|y - x| + \Delta(u - t)}{(u - t)^\beta}\right]^2 |t' - t|^{2\beta-1} \quad \text{for } \tfrac{1}{2} \leq \frac{u - t'}{u - t} \leq 2 \end{aligned} \tag{A.17}$$

and

(A.18) $|E(t, x'; u, y) - E(t, x; u, y)|$

$$\leqslant C\left[\frac{|y-x| + |y-x'| + \Delta(u-t)}{(u-t)^{\beta}}\right]^{1/\beta} |x'-x|^{2-1/\beta} \quad \textit{for } x' \in \mathbb{R}^N.$$

Proof. Note that the rôles of t and t' in (A.17) are essentially identical, likewise x and x' in (A.18): it therefore suffices to prove the inequalities without the modulus signs on the left-hand side. We may assume that $x = 0$ and $t = 0$. Then by a scaling (with $\sigma = 1/u$ and $\rho = 1/\Delta u$) we are reduced to the case where $u = 1$ and $\Delta = 1$. By Lemma A.3, any $\gamma^* \in \Gamma(0, 0; 1, y)$ satisfies

$$|\gamma_s^*| \leqslant C[|y| + 1]s^{\beta},$$

so we can apply Lemma A.5 with $K = C[|y| + 1]$ and get the desired result.

THEOREM A.7. *Let a and b/Δ be Hölder continuous in time of exponent $\alpha \in (0, 1]$ and constant $A \in [0, \infty)$, and let c/Δ^2 be Hölder continuous in space of exponent α and constant A/Δ^{α}. Then there is a constant $C(\alpha, \lambda) \in (0, \infty)$ such that, for all $t, u \in \mathbb{R}$ with $t < u$, and all $x, y \in \mathbb{R}^N$,*

(A.19) $|E(t', x; u, y) - E(t, x; u, y)|$

$$\leqslant C\left[\left(\frac{|y-x|}{u-t} + \Delta\right)(1 + A^{1/\alpha}(u-t))^{\frac{1}{2}}\right]^2 |t'-t| \quad \textit{for } \tfrac{1}{2} \leqslant \frac{u-t'}{u-t} \leqslant 2$$

and

(A.20) $|E(t, x'; u, y) - E(t, x; u, y)|$

$$\leqslant C\left[\left(\frac{|y-x|}{u-t} + \frac{|y-x'|}{u-t} + \Delta\right)(1 + A^{1/\alpha}(u-t))^{\frac{1}{2}}\right] |x'-x| \quad \textit{for } x' \in \mathbb{R}^N.$$

Proof. The proof is identical to that of Theorem A.6 except that we use Lemma A.4 to show that any $\gamma^* \in \Gamma(0, 0; 1, y)$ satisfies

$$|\gamma_s^*| \leqslant C[(|y| + 1)(1 + A^{1/\alpha})^{\frac{1}{2}}]s,$$

and take $K = C[(|y| + 1)(1 + A^{1/\alpha})^{\frac{1}{2}}]$ and $\beta = 1$ in Lemma A.5.

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