Exercises

2.3.1 Suppose S and T are independent exponential random variables of parameters α and β respectively. What is the distribution of $\min\{S,T\}$? What is the probability that $S \leq T$? Show that the two events $\{S < T\}$ and $\{\min\{S,T\} \geq t\}$ are independent.

2.3.2 Let $T_1, T_2, \dots$ be independent exponential random variables of parameter λ and let N be an independent geometric random variable with

$$\mathbb{P}(N = n) = \beta(1-\beta)^{n-1}, \qquad n = 1, 2, \dots .$$

Show that $T = \sum_{i=1}^{N} T_i$ has exponential distribution of parameter $\lambda\beta$.

2.3.3 Let $S_1, S_2, \dots$ be independent exponential random variables with parameters $\lambda_1, \lambda_2, \dots$ respectively. Show that $\lambda_1 S_1$ is exponential of parameter 1.

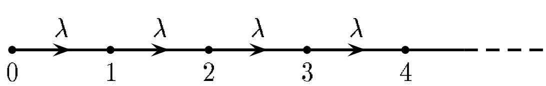
Use the strong law of large numbers to show, first in the special case $\lambda_n = 1$ for all n, and then subject only to the condition $\sup_n \lambda_n < \infty$, that

$$\mathbb{P}\left(\sum_{n=1}^{\infty} S_n = \infty\right) = 1 .$$

Is the condition $\sup_n \lambda_n < \infty$ absolutely necessary?

2.4 Poisson processes

Poisson processes are some of the simplest examples of continuous-time Markov chains. We shall also see that they may serve as building blocks for the most general continuous-time Markov chain. Moreover, a Poisson process is the natural probabilistic model for any uncoordinated stream of discrete events in continuous time. So we shall study Poisson processes first, both as a gentle warm-up for the general theory and because they are useful in themselves. The key result is Theorem 2.4.3, which provides three different descriptions of a Poisson process. The reader might well begin with the statement of this result and then see how it is used in the theorems and examples that follow. We shall begin with a definition in terms of jump chain and holding times (see Section 2.2). A right-continuous process $(X_t)_{t \geq 0}$ with values in $\{0, 1, 2, \dots\}$ is a *Poisson process of rate* λ $(0 < \lambda < \infty)$ if its holding times $S_1, S_2, \dots$ are independent exponential random variables of parameter λ and its jump chain is given by $Y_n = n$. Here is the diagram:

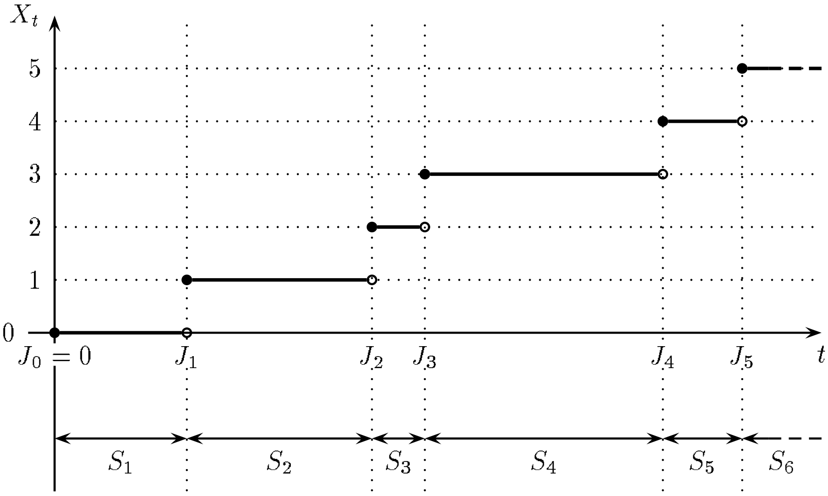


The associated Q-matrix is given by

$$Q = \begin{pmatrix} -\lambda & \lambda & & & \\ & -\lambda & \lambda & & \\ & & \ddots & \ddots & \\ & & & & \end{pmatrix}.$$

By Theorem 2.3.2 (or the strong law of large numbers) we have $\mathbb{P}(J_n \to \infty) = 1$ so there is no explosion and the law of $(X_t)_{t\geq 0}$ is uniquely determined. A simple way to construct a Poisson process of rate λ is to take a sequence $S_1, S_2, \dots$ of independent exponential random variables of parameter λ, to set $J_0 = 0$, $J_n = S_1 + \dots + S_n$ and then set

$$X_t = n \quad \text{if} \quad J_n \leq t < J_{n+1}.$$



The diagram illustrates a typical path. We now show how the memoryless property of the exponential holding times, Theorem 2.3.1, leads to a memoryless property of the Poisson process.

Theorem 2.4.1 (Markov property). *Let $(X_t)_{t\geq 0}$ be a Poisson process of rate λ. Then, for any $s \geq 0$, $(X_{s+t} - X_s)_{t\geq 0}$ is also a Poisson process of rate λ, independent of $(X_r : r \leq s)$.*

Proof. It suffices to prove the claim conditional on the event $X_s = i$, for each $i \geq 0$. Set $\widetilde{X}_t = X_{s+t} - X_s$. We have

$$\{X_s = i\} = \{J_i \leq s < J_{i+1}\} = \{J_i \leq s\} \cap \{S_{i+1} > s - J_i\}.$$

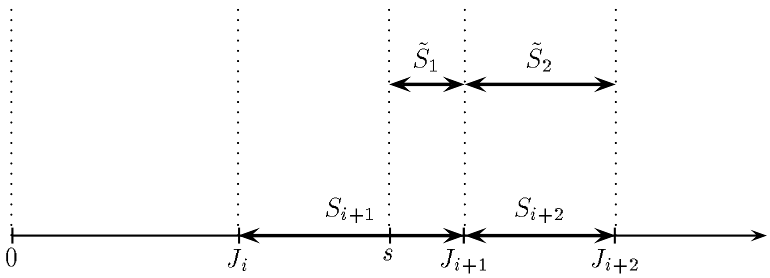
On this event

$$X_r = \sum_{j=1}^{i} 1_{\{S_j \le t\}} \quad \text{for } r \le s$$

and the holding times $\widetilde{S}_1, \widetilde{S}_2, \dots$ of $(\widetilde{X}_t)_{t \ge 0}$ are given by

$$\widetilde{S}_1 = S_{i+1} - (s - J_i), \quad \widetilde{S}_n = S_{i+n} \quad \text{for } n \ge 2$$

as shown in the diagram.



Recall that the holding times $S_1, S_2, \dots$ are independent $E(\lambda)$. Condition on $S_1, \dots, S_i$ and $\{X_s = i\}$, then by the memoryless property of S_{i+1} and independence, $\widetilde{S}_1, \widetilde{S}_2, \dots$ are themselves independent $E(\lambda)$. Hence, conditional on $\{X_s = i\}$, $\widetilde{S}_1, \widetilde{S}_2, \dots$ are independent $E(\lambda)$, and independent of $S_1, \dots, S_i$. Hence, conditional on $\{X_s = i\}$, $(\widetilde{X}_t)_{t \ge 0}$ is a Poisson process of rate λ and independent of $(X_r : r \le s)$. □

In fact, we shall see in Section 6.5, by an argument in essentially the same spirit that the result also holds with s replaced by any stopping time T of $(X_t)_{t \ge 0}$.

Theorem 2.4.2 (Strong Markov property). *Let $(X_t)_{t \ge 0}$ be a Poisson process of rate λ and let T be a stopping time of $(X_t)_{t \ge 0}$. Then, conditional on $T < \infty$, $(X_{T+t} - X_T)_{t \ge 0}$ is also a Poisson process of rate λ, independent of $(X_s : s \le T)$.*

Here is some standard terminology. If $(X_t)_{t \ge 0}$ is a real-valued process, we can consider its *increment* $X_t - X_s$ over any interval $(s, t]$. We say that $(X_t)_{t \ge 0}$ has *stationary* increments if the distribution of $X_{s+t} - X_s$ depends only on $t \ge 0$. We say that $(X_t)_{t \ge 0}$ has *independent* increments if its increments over any finite collection of disjoint intervals are independent.

We come to the key result for the Poisson process, which gives two conditions equivalent to the jump chain/holding time characterization which we took as our original definition. Thus we have three alternative definitions of the same process.

Theorem 2.4.3. *Let $(X_t)_{t\geq 0}$ be an increasing, right-continuous integer-valued process starting from 0. Let $0 < \lambda < \infty$. Then the following three conditions are equivalent:*

(a) (jump chain/holding time definition) *the holding times $S_1, S_2, \dots$ of $(X_t)_{t\geq 0}$ are independent exponential random variables of parameter λ and the jump chain is given by $Y_n = n$ for all n;*

(b) (infinitesimal definition) *$(X_t)_{t\geq 0}$ has independent increments and, as $h \downarrow 0$, uniformly in t,*

$$\mathbb{P}(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h), \quad \mathbb{P}(X_{t+h} - X_t = 1) = \lambda h + o(h);$$

(c) (transition probability definition) *$(X_t)_{t\geq 0}$ has stationary independent increments and, for each t, X_t has Poisson distribution of parameter λt.*

If $(X_t)_{t\geq 0}$ satisfies any of these conditions then it is called a *Poisson process of rate* λ.

Proof. (a) $\Rightarrow$ (b) If (a) holds, then, by the Markov property, for any $t, h \geq 0$, the increment $X_{t+h} - X_t$ has the same distribution as X_h and is independent of $(X_s : s \leq t)$. So $(X_t)_{t\geq 0}$ has independent increments and as $h \downarrow 0$

$$\begin{aligned}
\mathbb{P}(X_{t+h} - X_t \geq 1) &= \mathbb{P}(X_h \geq 1) = \mathbb{P}(J_1 \leq h) = 1 - e^{-\lambda h} = \lambda h + o(h), \\
\mathbb{P}(X_{t+h} - X_t \geq 2) &= \mathbb{P}(X_h \geq 2) = \mathbb{P}(J_2 \leq h) \\
&\leq \mathbb{P}(S_1 \leq h \text{ and } S_2 \leq h) = (1 - e^{-\lambda h})^2 = o(h),
\end{aligned}$$

which implies (b).

(b) $\Rightarrow$ (c) If (b) holds, then, for $i = 2, 3, \dots$, we have $\mathbb{P}(X_{t+h} - X_t = i) = o(h)$ as $h \downarrow 0$, uniformly in t. Set $p_j(t) = \mathbb{P}(X_t = j)$. Then, for $j = 1, 2, \dots$,

$$\begin{aligned}
p_j(t+h) = \mathbb{P}(X_{t+h} = j) &= \sum_{i=0}^{j} \mathbb{P}(X_{t+h} - X_t = i)\, \mathbb{P}(X_t = j - i) \\
&= (1 - \lambda h + o(h)) p_j(t) + (\lambda h + o(h)) p_{j-1}(t) + o(h)
\end{aligned}$$

so

$$\frac{p_j(t+h) - p_j(t)}{h} = -\lambda p_j(t) + \lambda p_{j-1}(t) + O(h).$$

Since this estimate is uniform in t we can put $t = s - h$ to obtain for all $s \geq h$

$$\frac{p_j(s) - p_j(s-h)}{h} = -\lambda p_j(s-h) + \lambda p_{j-1}(s-h) + O(h).$$

Now let $h \downarrow 0$ to see that $p_j(t)$ is first continuous and then differentiable and satisfies the differential equation

$$p_j'(t) = -\lambda p_j(t) + \lambda p_{j-1}(t).$$

By a simpler argument we also find

$$p_0'(t) = -\lambda p_0(t).$$

Since $X_0 = 0$ we have initial conditions

$$p_0(0) = 1, \quad p_j(0) = 0 \quad \text{for } j = 1, 2, \dots .$$

As we saw in Example 2.1.4, this system of equations has a unique solution given by

$$p_j(t) = e^{-\lambda t}\frac{(\lambda t)^j}{j!}, \qquad j = 0, 1, 2, \dots .$$

Hence $X_t \sim P(\lambda t)$. If $(X_t)_{t\geq 0}$ satisfies (b), then certainly $(X_t)_{t\geq 0}$ has independent increments, but also $(X_{s+t} - X_s)_{t\geq 0}$ satisfies (b), so the above argument shows $X_{s+t} - X_s \sim P(\lambda t)$, for any s, which implies (c).

(c) $\Rightarrow$ (a) There is a process satisfying (a) and we have shown that it must then satisfy (c). But condition (c) determines the finite-dimensional distributions of $(X_t)_{t\geq 0}$ and hence the distribution of jump chain and holding times. So if one process satisfying (c) also satisfies (a), so must every process satisfying (c). □

The differential equations which appeared in the proof are really the forward equations for the Poisson process. To make this clear, consider the possibility of starting the process from i at time 0, writing $\mathbb{P}_i$ as a reminder, and set

$$p_{ij}(t) = \mathbb{P}_i(X_t = j).$$

Then, by spatial homogeneity $p_{ij}(t) = p_{j-i}(t)$, and we could rewrite the differential equations as

$$\begin{aligned} p_{i0}'(t) &= -\lambda p_{i0}(t), & p_{i0}(0) &= \delta_{i0}, \\ p_{ij}'(t) &= \lambda p_{i,j-1}(t) - \lambda p_{ij}(t), & p_{ij}(0) &= \delta_{ij} \end{aligned}$$

or, in matrix form, for Q as above,

$$P'(t) = P(t)Q, \qquad P(0) = I.$$

Theorem 2.4.3 contains a great deal of information about the Poisson process of rate λ. It can be useful when trying to decide whether a given process is a Poisson process as it gives you three alternative conditions to check, and it is likely that one will be easier to check than another. On the other hand it can also be useful when answering a question about a given Poisson process as this question may be more closely connected to one definition than another. For example, you might like to consider the difficulties in approaching the next result using the jump chain/holding time definition.

Theorem 2.4.4. *If $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are independent Poisson processes of rates λ and μ, respectively, then $(X_t+Y_t)_{t\geq 0}$ is a Poisson process of rate $\lambda+\mu$.*

Proof. We shall use the infinitesimal definition, according to which $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ have independent increments and, as $h\downarrow 0$, uniformly in t,

$$\mathbb{P}(X_{t+h}-X_t=0)=1-\lambda h+o(h),\quad \mathbb{P}(X_{t+h}-X_t=1)=\lambda h+o(h),$$
$$\mathbb{P}(Y_{t+h}-Y_t=0)=1-\mu h+o(h),\quad \mathbb{P}(Y_{t+h}-Y_t=1)=\mu h+o(h).$$

Set $Z_t=X_t+Y_t$. Then, since $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are independent, $(Z_t)_{t\geq 0}$ has independent increments and, as $h\downarrow 0$, uniformly in t,

$$\begin{aligned}\mathbb{P}(Z_{t+h}-Z_t=0)&=\mathbb{P}(X_{t+h}-X_t=0)\mathbb{P}(Y_{t+h}-Y_t=0)\\&=(1-\lambda h+o(h))(1-\mu h+o(h))=1-(\lambda+\mu)h+o(h),\\ \mathbb{P}(Z_{t+h}-Z_t=1)&=\mathbb{P}(X_{t+h}-X_t=1)\mathbb{P}(Y_{t+h}-Y_t=0)\\&\quad+\mathbb{P}(X_{t+h}-X_t=0)\mathbb{P}(Y_{t+h}-Y_t=1)\\&=(\lambda h+o(h))(1-\mu h+o(h))+(1-\lambda h+o(h))(\mu h+o(h))\\&=(\lambda+\mu)h+o(h).\end{aligned}$$

Hence $(Z_t)_{t\geq 0}$ is a Poisson process of rate $\lambda+\mu$. □

Next we establish some relations between Poisson processes and the uniform distribution. Notice that the conclusions are independent of the rate of the process considered. The results say in effect that the jumps of a Poisson process are as randomly distributed as possible.

Theorem 2.4.5. *Let $(X_t)_{t\geq 0}$ be a Poisson process. Then, conditional on $(X_t)_{t\geq 0}$ having exactly one jump in the interval $[s,s+t]$, the time at which that jump occurs is uniformly distributed on $[s,s+t]$.*

Proof. We shall use the finite-dimensional distribution definition. By stationarity of increments, it suffices to consider the case $s=0$. Then, for $0\leq u\leq t$,

$$\begin{aligned}&\mathbb{P}(J_1\leq u\mid X_t=1)=\mathbb{P}(J_1\leq u\text{ and }X_t=1)/\mathbb{P}(X_t=1)\\&\quad=\mathbb{P}(X_u=1\text{ and }X_t-X_u=0)/\mathbb{P}(X_t=1)\\&\quad=\lambda ue^{-\lambda u}e^{-\lambda(t-u)}/(\lambda te^{-\lambda t})=u/t.\end{aligned}$$

□

Theorem 2.4.6. *Let $(X_t)_{t\geq 0}$ be a Poisson process. Then, conditional on the event $\{X_t=n\}$, the jump times $J_1,\dots,J_n$ have joint density function*

$$f(t_1,\dots,t_n)=n!\,1_{\{0\leq t_1\leq\dots\leq t_n\leq t\}}.$$

Thus, conditional on $\{X_t=n\}$, the jump times $J_1,\dots,J_n$ have the same distribution as an ordered sample of size n from the uniform distribution on $[0,t]$.

Proof. The holding times $S_1,\dots,S_{n+1}$ have joint density function

$$\lambda^{n+1}e^{-\lambda(s_1+\dots+s_{n+1})}1_{\{s_1,\dots,s_{n+1}\geq 0\}}$$

so the jump times $J_1, \dots, J_{n+1}$ have joint density function

$$\lambda^{n+1} e^{-\lambda t_{n+1}} 1_{\{0 \le t_1 \le \dots \le t_{n+1}\}}.$$

So for $A \subseteq \mathbb{R}^n$ we have

$$\mathbb{P}\big((J_1, \dots, J_n) \in A \text{ and } X_t = n\big) = \mathbb{P}\big((J_1, \dots, J_n) \in A \text{ and } J_n \le t < J_{n+1}\big)$$
$$= e^{-\lambda t} \lambda^n \int_{(t_1, \dots, t_n) \in A} 1_{\{0 \le t_1 \le \dots \le t_n \le t\}} dt_1 \dots dt_n$$

and since $\mathbb{P}(X_t = n) = e^{-\lambda t}\lambda^n / n!$ we obtain

$$\mathbb{P}\big((J_1, \dots, J_n) \in A \mid X_t = n\big) = \int_A f(t_1, \dots, t_n) dt_1 \dots dt_n$$

as required. □

We finish with a simple example typical of many problems making use of a range of properties of the Poisson process.

Example 2.4.7

Robins and blackbirds make brief visits to my birdtable. The probability that in any small interval of duration h a robin will arrive is found to be $\rho h + o(h)$, whereas the corresponding probability for blackbirds is $\beta h + o(h)$. What is the probability that the first two birds I see are both robins? What is the distribution of the total number of birds seen in time t? Given that this number is n, what is the distribution of the number of blackbirds seen in time t?

By the infinitesimal characterization, the number of robins seen by time t is a Poisson process $(R_t)_{t \ge 0}$ of rate ρ, and the number of blackbirds is a Poisson process $(B_t)_{t \ge 0}$ of rate β. The times spent waiting for the first robin or blackbird are independent exponential random variables S_1 and T_1 of parameters ρ and β respectively. So a robin arrives first with probability $\rho/(\rho + \beta)$ and, by the memoryless property of T_1, the probability that the first two birds are robins is $\rho^2/(\rho + \beta)^2$. By Theorem 2.4.4 the total number of birds seen in an interval of duration t has Poisson distribution of parameter $(\rho + \beta)t$. Finally

$$\mathbb{P}(B_t = k \mid R_t + B_t = n) = \mathbb{P}(B_t = k \text{ and } R_t = n - k)/\mathbb{P}(R_t + B_t = n)$$
$$= \left(\frac{e^{-\beta}\beta^k}{k!}\right)\left(\frac{e^{-\rho}\rho^{n-k}}{(n-k)!}\right) \Big/ \left(\frac{e^{-(\rho+\beta)}(\rho + \beta)^n}{n!}\right)$$
$$= \binom{n}{k}\left(\frac{\beta}{\rho + \beta}\right)^k \left(\frac{\rho}{\rho + \beta}\right)^{n-k}$$

so if n birds are seen in time t, then the distribution of the number of blackbirds is binomial of parameters n and $\beta/(\rho + \beta)$.

Exercises

2.4.1 State the transition probability definition of a Poisson process. Show directly from this definition that the first jump time J_1 of a Poisson process of rate λ is exponential of parameter λ.

Show also (from the same definition and without assuming the strong Markov property) that

$$\mathbb{P}(t_1 < J_1 \leq t_2 < J_2) = e^{-\lambda t_1}\lambda(t_2 - t_1)e^{-\lambda(t_2-t_1)}$$

and hence that $J_2 - J_1$ is also exponential of parameter λ and independent of J_1.

2.4.2 Show directly from the infinitesimal definition that the first jump time J_1 of a Poisson process of rate λ has exponential distribution of parameter λ.

2.4.3 Arrivals of the Number 1 bus form a Poisson process of rate one bus per hour, and arrivals of the Number 7 bus form an independent Poisson process of rate seven buses per hour.

(a) What is the probability that exactly three buses pass by in one hour?

(b) What is the probability that exactly three Number 7 buses pass by while I am waiting for a Number 1?

(c) When the maintenance depot goes on strike half the buses break down before they reach my stop. What, then, is the probability that I wait for 30 minutes without seeing a single bus?

2.4.4 A radioactive source emits particles in a Poisson process of rate λ. The particles are each emitted in an independent random direction. A Geiger counter placed near the source records a fraction p of the particles emitted. What is the distribution of the number of particles recorded in time t?

2.4.5 A pedestrian wishes to cross a single lane of fast-moving traffic. Suppose the number of vehicles that have passed by time t is a Poisson process of rate λ, and suppose it takes time a to walk across the lane. Assuming that the pedestrian can foresee correctly the times at which vehicles will pass by, how long on average does it take to cross over safely? [*Consider the time at which the first car passes.*]

How long on average does it take to cross two similar lanes (a) when one must walk straight across (assuming that the pedestrian will not cross if, at any time whilst crossing, a car would pass in either direction), (b) when an island in the middle of the road makes it safe to stop half-way?

2.5 Birth processes

A birth process is a generalization of a Poisson process in which the parameter λ is allowed to depend on the current state of the process. The data for a birth process consist of *birth rates* $0 \leq q_j < \infty$, where $j = 0, 1, 2, \ldots$. We begin with a definition in terms of jump chain and holding times. A minimal right-continuous process $(X_t)_{t\geq 0}$ with values in $\{0, 1, 2, \ldots\} \cup \{\infty\}$ is a *birth process of rates* $(q_j : j \geq 0)$ if, conditional on $X_0 = i$, its holding times $S_1, S_2, \ldots$ are independent exponential