1.3 Hitting times and absorption probabilities

Let $(X_n)_{n\geq 0}$ be a Markov chain with transition matrix P. The *hitting time* of a subset A of I is the random variable $H^A : \Omega \to \{0,1,2,\dots\} \cup \{\infty\}$ given by

$$H^A(\omega) = \inf\{n \geq 0 : X_n(\omega) \in A\}$$

where we agree that the infimum of the empty set $\emptyset$ is ∞. The probability starting from i that $(X_n)_{n\geq 0}$ ever hits A is then

$$h_i^A = P_i(H^A < \infty).$$

When A is a closed class, h_i^A is called the *absorption probability*. The mean time taken for $(X_n)_{n\geq 0}$ to reach A is given by

$$k_i^A = E_i(H^A) = \sum_{n<\infty} nP(H^A = n) + \infty P(H^A = \infty).$$

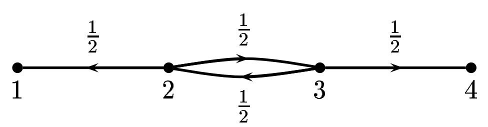
We shall often write less formally

$$h_i^A = P_i(\text{hit } A), \quad k_i^A = E_i(\text{time to hit } A).$$

Remarkably, these quantities can be calculated explicitly by means of certain linear equations associated with the transition matrix P. Before we give the general theory, here is a simple example.

Example 1.3.1

Consider the chain with the following diagram:



Starting from 2, what is the probability of absorption in 4? How long does it take until the chain is absorbed in 1 or 4?

Introduce

$$h_i = P_i(\text{hit } 4), \quad k_i = E_i(\text{time to hit } \{1,4\}).$$

Clearly, $h_1 = 0$, $h_4 = 1$ and $k_1 = k_4 = 0$. Suppose now that we start at 2, and consider the situation after making one step. With probability 1/2 we jump to 1 and with probability 1/2 we jump to 3. So

$$h_2 = \tfrac{1}{2}h_1 + \tfrac{1}{2}h_3, \quad k_2 = 1 + \tfrac{1}{2}k_1 + \tfrac{1}{2}k_3.$$

The 1 appears in the second formula because we count the time for the first step. Similarly,

$$h_3 = \tfrac{1}{2}h_2 + \tfrac{1}{2}h_4, \quad k_3 = 1 + \tfrac{1}{2}k_2 + \tfrac{1}{2}k_4.$$

Hence

$$h_2 = \tfrac{1}{2}h_3 = \tfrac{1}{2}(\tfrac{1}{2}h_2 + \tfrac{1}{2}),$$
$$k_2 = 1 + \tfrac{1}{2}k_3 = 1 + \tfrac{1}{2}(1 + \tfrac{1}{2}k_2).$$

So, starting from 2, the probability of hitting 4 is 1/3 and the mean time to absorption is 2. Note that in writing down the first equations for h_2 and k_2 we made implicit use of the Markov property, in assuming that the chain begins afresh from its new position after the first jump. Here is a general result for hitting probabilities.

Theorem 1.3.2. *The vector of hitting probabilities $h^A = (h_i^A : i \in I)$ is the minimal non-negative solution to the system of linear equations*

$$\begin{cases} h_i^A = 1 & \text{for } i \in A \\ h_i^A = \sum_{j \in I} p_{ij} h_j^A & \text{for } i \notin A. \end{cases} \tag{1.3}$$

(Minimality means that if $x = (x_i : i \in I)$ is another solution with $x_i \geq 0$ for all i, then $x_i \geq h_i$ for all i.)

Proof. First we show that h^A satisfies (1.3). If $X_0 = i \in A$, then $H^A = 0$, so $h_i^A = 1$. If $X_0 = i \notin A$, then $H^A \geq 1$, so by the Markov property

$$P_i(H^A < \infty \mid X_1 = j) = P_j(H^A < \infty) = h_j^A$$

and

$$\begin{aligned} h_i^A = P_i(H^A < \infty) &= \sum_{j \in I} P_i(H^A < \infty, X_1 = j) \\ &= \sum_{j \in I} P_i(H^A < \infty \mid X_1 = j) P_i(X_1 = j) = \sum_{j \in I} p_{ij} h_j^A. \end{aligned}$$

Suppose now that $x = (x_i : i \in I)$ is any solution to (1.3). Then $h_i^A = x_i = 1$ for $i \in A$. Suppose $i \notin A$, then

$$x_i = \sum_{j \in I} p_{ij} x_j = \sum_{j \in A} p_{ij} + \sum_{j \notin A} p_{ij} x_j.$$

Substitute for x_j to obtain

$$\begin{aligned} x_i &= \sum_{j \in A} p_{ij} + \sum_{j \notin A} p_{ij} \left(\sum_{k \in A} p_{jk} + \sum_{k \notin A} p_{jk} x_k \right) \\ &= P_i(X_1 \in A) + P_i(X_1 \notin A, X_2 \in A) + \sum_{j \notin A} \sum_{k \notin A} p_{ij} p_{jk} x_k. \end{aligned}$$

By repeated substitution for x in the final term we obtain after n steps

$$\begin{aligned} x_i = P_i(X_1 \in A) + \ldots &+ P_i(X_1 \notin A, \ldots, X_{n-1} \notin A, X_n \in A) \\ &+ \sum_{j_1 \notin A} \cdots \sum_{j_n \notin A} p_{ij_1} p_{j_1 j_2} \ldots p_{j_{n-1} j_n} x_{j_n}. \end{aligned}$$

Now if x is non-negative, so is the last term on the right, and the remaining terms sum to $P_i(H^A \leq n)$. So $x_i \geq P_i(H^A \leq n)$ for all n and then

$$x_i \geq \lim_{n \to \infty} P_i(H^A \leq n) = P_i(H^A < \infty) = h_i. \qquad \square$$

Example 1.3.1 (continued)

The system of linear equations (1.3) for $h = h^{\{4\}}$ are given here by

$$h_4 = 1,$$
$$h_2 = \tfrac{1}{2}h_1 + \tfrac{1}{2}h_3,\ h_3 = \tfrac{1}{2}h_2 + \tfrac{1}{2}h_4$$

so that

$$h_2 = \tfrac{1}{2}h_1 + \tfrac{1}{2}(\tfrac{1}{2}h_2 + \tfrac{1}{2})$$

and

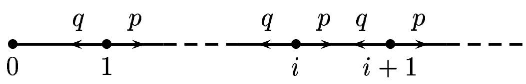
$$h_2 = \tfrac{1}{3} + \tfrac{2}{3}h_1,\ h_3 = \tfrac{2}{3} + \tfrac{1}{3}h_1.$$

The value of h_1 is not determined by the system (1.3), but the minimality condition now makes us take $h_1 = 0$, so we recover $h_2 = 1/3$ as before. Of course, the extra boundary condition $h_1 = 0$ was obvious from the beginning so we built it into our system of equations and did not have to worry about minimal non-negative solutions.

In cases where the state-space is infinite it may not be possible to write down a corresponding extra boundary condition. Then, as we shall see in the next examples, the minimality condition is essential.

Example 1.3.3 (Gamblers' ruin)

Consider the Markov chain with diagram



where $0 < p = 1 - q < 1$. The transition probabilities are

$$p_{00} = 1,$$
$$p_{i,i-1} = q,\ p_{i,i+1} = p \quad \text{for } i = 1, 2, \ldots.$$

Imagine that you enter a casino with a fortune of £i and gamble, £1 at a time, with probability p of doubling your stake and probability q of losing it. The resources of the casino are regarded as infinite, so there is no upper limit to your fortune. But what is the probability that you leave broke?

Set $h_i = P_i(\text{hit } 0)$, then h is the minimal non-negative solution to

$$h_0 = 1,$$
$$h_i = ph_{i+1} + qh_{i-1}, \quad \text{for } i = 1, 2, \ldots.$$

If $p \neq q$ this recurrence relation has a general solution

$$h_i = A + B\left(\frac{q}{p}\right)^i.$$

(See Section 1.11.) If $p < q$, which is the case in most successful casinos, then the restriction $0 \leq h_i \leq 1$ forces $B = 0$, so $h_i = 1$ for all i. If $p > q$, then since $h_0 = 1$ we get a family of solutions

$$h_i = \left(\frac{q}{p}\right)^i + A\left(1 - \left(\frac{q}{p}\right)^i\right);$$

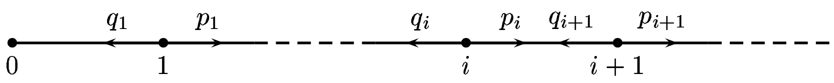
for a non-negative solution we must have $A \geq 0$, so the minimal non-negative solution is $h_i = (q/p)^i$. Finally, if $p = q$ the recurrence relation has a general solution

$$h_i = A + Bi$$

and again the restriction $0 \leq h_i \leq 1$ forces $B = 0$, so $h_i = 1$ for all i. Thus, even if you find a fair casino, you are certain to end up broke. This apparent paradox is called gamblers' ruin.

Example 1.3.4 (Birth-and-death chain)

Consider the Markov chain with diagram



where, for $i = 1, 2, \ldots$, we have $0 < p_i = 1 - q_i < 1$. As in the preceding example, 0 is an absorbing state and we wish to calculate the absorption probability starting from i. But here we allow p_i and q_i to depend on i.

Such a chain may serve as a model for the size of a population, recorded each time it changes, p_i being the probability that we get a birth before a death in a population of size i. Then $h_i = P_i(\text{hit } 0)$ is the extinction probability starting from i.

We write down the usual system of equations

$$\begin{aligned} h_0 &= 1, \\ h_i &= p_i h_{i+1} + q_i h_{i-1}, \quad \text{for } i = 1, 2, \ldots. \end{aligned}$$

This recurrence relation has variable coefficients so the usual technique fails. But consider $u_i = h_{i-1} - h_i$, then $p_i u_{i+1} = q_i u_i$, so

$$u_{i+1} = \left(\frac{q_i}{p_i}\right) u_i = \left(\frac{q_i q_{i-1} \ldots q_1}{p_i p_{i-1} \ldots p_1}\right) u_1 = \gamma_i u_1$$

where the final equality defines γ_i. Then

$$u_1 + \ldots + u_i = h_0 - h_i$$

so

$$h_i = 1 - A(\gamma_0 + \ldots + \gamma_{i-1})$$

where $A = u_1$ and $\gamma_0 = 1$. At this point A remains to be determined. In the case $\sum_{i=0}^{\infty} \gamma_i = \infty$, the restriction $0 \leq h_i \leq 1$ forces $A = 0$ and $h_i = 1$ for all i. But if $\sum_{i=0}^{\infty} \gamma_i < \infty$ then we can take $A > 0$ so long as

$$1 - A(\gamma_0 + \ldots + \gamma_{i-1}) \geq 0 \quad \text{for all } i.$$

Thus the minimal non-negative solution occurs when $A = \left(\sum_{i=0}^{\infty} \gamma_i\right)^{-1}$ and then

$$h_i = \sum_{j=i}^{\infty} \gamma_j \bigg/ \sum_{j=0}^{\infty} \gamma_j .$$

In this case, for $i = 1, 2, \ldots$, we have $h_i < 1$, so the population survives with positive probability.

Here is the general result on mean hitting times. Recall that $k_i^A = E_i(H^A)$, where H^A is the first time $(X_n)_{n \geq 0}$ hits A. We use the notation 1_B for the indicator function of B, so, for example, $1_{X_1 = j}$ is the random variable equal to 1 if $X_1 = j$ and equal to 0 otherwise.

Theorem 1.3.5. *The vector of mean hitting times $k^A = (k^A : i \in I)$ is the minimal non-negative solution to the system of linear equations*

$$\begin{cases} k_i^A = 0 & \text{for } i \in A \\ k_i^A = 1 + \sum_{j \notin A} p_{ij} k_j^A & \text{for } i \notin A. \end{cases} \tag{1.4}$$

Proof. First we show that k^A satisfies (1.4). If $X_0 = i \in A$, then $H^A = 0$, so $k_i^A = 0$. If $X_0 = i \notin A$, then $H^A \geq 1$, so, by the Markov property,

$$E_i(H^A \mid X_1 = j) = 1 + E_j(H^A)$$

and

$$\begin{aligned} k_i^A = E_i(H^A) &= \sum_{j \in I} E_i(H^A 1_{X_1 = j}) \\ &= \sum_{j \in I} E_i(H^A \mid X_1 = j) P_i(X_1 = j) = 1 + \sum_{j \notin A} p_{ij} k_j^A . \end{aligned}$$

Suppose now that $y = (y_i : i \in I)$ is any solution to (1.4). Then $k_i^A = y_i = 0$ for $i \in A$. If $i \notin A$, then

$$\begin{aligned} y_i &= 1 + \sum_{j \notin A} p_{ij} y_j \\ &= 1 + \sum_{j \notin A} p_{ij} \left(1 + \sum_{k \notin A} p_{jk} y_k\right) \\ &= P_i(H^A \geq 1) + P_i(H^A \geq 2) + \sum_{j \notin A} \sum_{k \notin A} p_{ij} p_{jk} y_k . \end{aligned}$$

By repeated substitution for y in the final term we obtain after n steps

$$y_i = P_i(H^A \geq 1) + \ldots + P_i(H^A \geq n) + \sum_{j_1 \notin A} \cdots \sum_{j_n \notin A} p_{ij_1} p_{j_1 j_2} \cdots p_{j_{n-1} j_n} y_{j_n}.$$

So, if y is non-negative,

$$y_i \geq P_i(H^A \geq 1) + \ldots + P_i(H^A \geq n)$$

and, letting $n \to \infty$,

$$y_i \geq \sum_{n=1}^{\infty} P_i(H^A \geq n) = E_i(H^A) = x_i. \qquad \square$$

Exercises

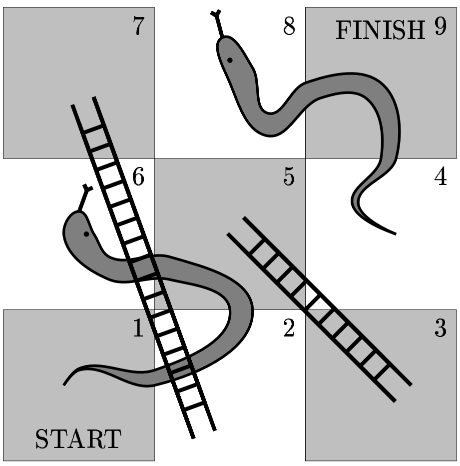
1.3.1 Prove the claims (a), (b) and (c) made in example (v) of the Introduction.

1.3.2 A gambler has £2 and needs to increase it to £10 in a hurry. He can play a game with the following rules: a fair coin is tossed; if a player bets on the right side, he wins a sum equal to his stake, and his stake is returned; otherwise he loses his stake. The gambler decides to use a bold strategy in which he stakes all his money if he has £5 or less, and otherwise stakes just enough to increase his capital, if he wins, to £10.

Let $X_0 = 2$ and let X_n be his capital after n throws. Prove that the gambler will achieve his aim with probability $1/5$.

What is the expected number of tosses until the gambler either achieves his aim or loses his capital?

1.3.3 A simple game of 'snakes and ladders' is played on a board of nine squares.



At each turn a player tosses a fair coin and advances one or two places according to whether the coin lands heads or tails. If you land at the foot of a ladder you climb to the top, but if you land at the head of a snake you slide down to the tail. How many turns on average does it take to complete the game?

What is the probability that a player who has reached the middle square will complete the game without slipping back to square 1?

1.3.4 Let $(X_n)_{n\geq 0}$ be a Markov chain on $\{0, 1, \dots\}$ with transition probabilities given by

$$p_{01} = 1, \quad p_{i,i+1} + p_{i,i-1} = 1, \quad p_{i,i+1} = \left(\frac{i+1}{i}\right)^2 p_{i,i-1}, \quad i \geq 1 \, .$$

Show that if $X_0 = 0$ then the probability that $X_n \geq 1$ for all $n \geq 1$ is $6/\pi^2$.