1.11 Appendix: recurrence relations

Recurrence relations often arise in the linear equations associated to Markov chains. Here is an account of the simplest cases. A more specialized case was dealt with in Example 1.3.4. In Example 1.1.4 we found a recurrence relation of the form

\[ x_{n+1} = ax_n + b. \]

We look first for a constant solution \( x_n = x \); then \( x = ax + b \), so provided \( a \neq 1 \) we must have \( x = b/(1 - a) \). Now \( y_n = x_n - b/(1 - a) \) satisfies \( y_{n+1} = ay_n \), so \( y_n = a^n y_0 \). Thus the general solution when \( a \neq 1 \) is given by

\[ x_n = Aa^n + b/(1 - a) \]

where \( A \) is a constant. When \( a = 1 \) the general solution is obviously

\[ x_n = x_0 + nb. \]

In Example 1.3.3 we found a recurrence relation of the form

\[ ax_{n+1} + bx_n + cx_{n-1} = 0 \]

where \( a \) and \( c \) were both non-zero. Let us try a solution of the form \( x_n = \lambda^n \); then

\[ a\lambda^2 + b\lambda + c = 0. \]

Denote by \( \alpha \) and \( \beta \) the roots of this quadratic. Then

\[ y_n = A\alpha^n + B\beta^n \]

is a solution. If \( \alpha \neq \beta \) then we can solve the equations

\[ x_0 = A + B, \quad x_1 = A\alpha + B\beta \]

so that \( y_0 = x_0 \) and \( y_1 = x_1 \); but

\[ a(y_{n+1} - x_{n+1}) + b(y_n - x_n) + c(y_{n-1} - x_{n-1}) = 0 \]

for all \( n \), so by induction \( y_n = x_n \) for all \( n \). If \( \alpha = \beta \neq 0 \), then

\[ y_n = (A + nB)\alpha^n \]

is a solution and we can solve

\[ x_0 = A\alpha^n, \quad x_1 = (A + B)\alpha^n \]

so that \( y_0 = x_0 \) and \( y_1 = x_1 \); then, by the same argument, \( y_n = x_n \) for all \( n \). The case \( \alpha = \beta = 0 \) does not arise. Hence the general solution is given by

\[ x_n = \begin{cases} 
A\alpha^n + B\beta^n & \text{if } \alpha \neq \beta \\
(A + nB)\alpha^n & \text{if } \alpha = \beta.
\end{cases} \]
1.12 Appendix: asymptotics for $n!$

Our analysis of recurrence and transience for random walks in Section 1.6 rested heavily on the use of the asymptotic relation

$$n! \sim A \sqrt{n} (n/e)^n \quad \text{as } n \to \infty$$

for some $A \in [1, \infty)$. Here is a derivation.

We make use of the power series expansions for $|t| < 1$

$$\log(1 + t) = t - \frac{1}{2} t^2 + \frac{1}{3} t^3 - \ldots$$
$$\log(1 - t) = -t - \frac{1}{2} t^2 - \frac{1}{3} t^3 - \ldots$$

By subtraction we obtain

$$\frac{1}{2} \log \left( \frac{1 + t}{1 - t} \right) = t + \frac{1}{3} t^3 + \frac{1}{5} t^5 + \ldots$$

Set $A_n = n!/(n^{n+1/2} e^{-n})$ and $a_n = \log A_n$. Then, by a straightforward calculation

$$a_n - a_{n+1} = (2n + 1) \frac{1}{2} \log \left( \frac{1 + (2n + 1)^{-1}}{1 - (2n + 1)^{-1}} \right) - 1.$$

By the series expansion written above we have

$$a_n - a_{n+1} = (2n + 1) \left( \frac{1}{(2n + 1)} + \frac{1}{3 (2n + 1)^3} + \frac{1}{5 (2n + 1)^5} + \ldots \right) - 1$$
$$= \frac{1}{3} \frac{1}{(2n + 1)^2} + \frac{1}{5} \frac{1}{(2n + 1)^4} + \ldots$$
$$\leq \frac{1}{3} \left\{ \frac{1}{(2n + 1)^2} + \frac{1}{(2n + 1)^4} + \ldots \right\}$$
$$= \frac{1}{3} \frac{1}{(2n + 1)^2 - 1} = \frac{1}{12n} - \frac{1}{12(n + 1)}.$$

It follows that $a_n$ decreases and $a_n - 1/(12n)$ increases as $n \to \infty$. Hence $a_n \to a$ for some $a \in [0, \infty)$ and hence $A_n \to A$, as $n \to \infty$, where $A = e^a$. 