MATHEMATICAL TRIPOS: PART IB **Michaelmas Term 2005**

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Example Sheet 1

1. Let $B_1, B_2, \ldots$ be disjoint events with $\bigcup_{n=1}^{\infty} B_n = \Omega$. Show that if A is another event and $\mathbb{P}(A|B_n) = p$ for all n then $\mathbb{P}(A) = p$. Deduce that if X and Y are discrete random variables then the following are equivalent:

(a) X and Y are independent,
(b) the conditional distribution of X given $Y = y$ is independent of y.

2. Show that if $(X_n)_{n\geq 0}$ is a discrete-time Markov chain with transition matrix P and $Y_n = X_{kn}$, then $(Y_n)_{n\geq 0}$ is a Markov chain with transition matrix P^k.

3. Let X_0 be a random variable with values in a countable set I. Let $Y_1, Y_2, \ldots$ be a sequence of independent random variables, uniformly distributed on $[0, 1]$. Suppose we are given a function

$$G : I \times [0, 1] \to I$$

and define inductively for $n \geq 0$

$$X_{n+1} = G(X_n, Y_{n+1}).$$

Show that $(X_n)_{n\geq 0}$ is a Markov chain and express its transition matrix P in terms of G. Can all Markov chains be realized in this way? How would you simulate a Markov chain using a computer?

4. Suppose that $Z_0, Z_1, \ldots$ are independent, identically distributed random variables such that $Z_i = 1$ with probability p and $Z_i = 0$ with probability $1 - p$. Set $S_0 = 0$, $S_n = Z_1 + \ldots + Z_n$. In each of the following cases determine whether $(X_n)_{n\geq 0}$ is a Markov chain:

(a) $X_n = Z_n$, (b) $X_n = S_n$,
(c) $X_n = S_0 + \ldots + S_n$, (d) $X_n = (S_n, S_0 + \ldots + S_n)$.

In the cases where $(X_n)_{n\geq 0}$ is a Markov chain find its state-space and transition matrix, and in the cases where it is not a Markov chain give an example where $P(X_{n+1} = i|X_n = j, X_{n-1} = k)$ is not independent of k.

5. A flea hops randomly on vertices of a triangle, hopping to each of the other vertices with equal probability. Find the probability that after n hops the flea is back where it started.

A second flea also hops about on the vertices of a triangle, but this flea is twice as likely to jump clockwise as anticlockwise. What is the probability that after n hops this second flea is back where it started? [*Hint*: $\frac{1}{2} \pm \frac{i}{2\sqrt{3}} = \frac{1}{\sqrt{3}} e^{\pm i\pi/6}$.]

6. A die is 'fixed' so that each time it is rolled the score cannot be the same as the preceding score, all other scores having probability $1/5$. If the first score is 6, what is the probability p that the $(n+1)$st score is 6? What is the probability that the $(n+1)$st score is 1?

7. Let $(X_n)_{n\geq 0}$ be a Markov chain on $\{1,2,3\}$ with transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ p & 1-p & 0 \end{pmatrix}.$$

Calculate $\mathbb{P}(X_n = 1 | X_0 = 1)$ when (a) $p = \frac{1}{16}$, (b) $p = \frac{1}{6}$, (c) $p = \frac{1}{12}$.

8. Identify the communicating classes of the transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

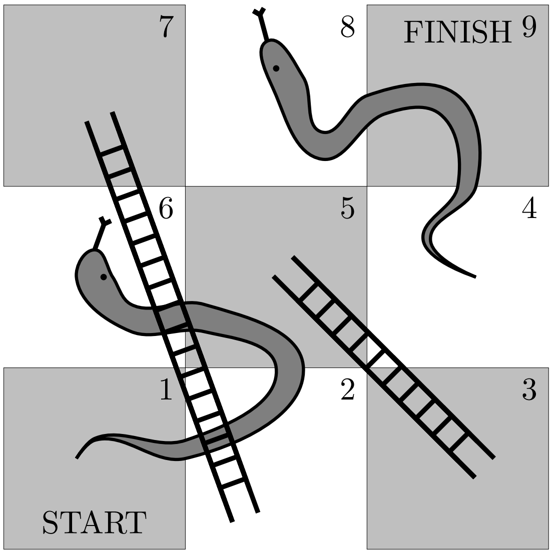
Which of the classes are closed?

9. Show that every transition matrix on a finite state-space has at least one closed communicating class. Find an example of a transition matrix with no closed communicating class.

10. A gambler has £2 and needs to increase it to £10 in a hurry. He can play a game with the following rules: a fair coin is tossed; if a player bets on the right side, he wins a sum equal to his stake, and his stake is returned; otherwise he loses his stake. The gambler decides to use a bold strategy in which he stakes all his money if he has £5 or less, and otherwise stakes just enough to increase his capital, if he wins, to £10. Let $X_0 = 2$ and let X_n be his capital after n throws. Prove that the gambler will achieve his aim with probability $1/5$.

What is the expected number of tosses until the gambler either achieves his aim or loses his capital?

11. A simple game of 'snakes and ladders' is played on a board of nine squares



At each turn a player tosses a fair coin and advances one or two places in the usual fashion, according to whether the coin lands heads or tails. How many turns on average does it take to complete the game?

What is the probability that a player who has reached the middle square will complete the game without slipping back to square 1?

12. Let $(X_n)_{n\geq 0}$ be a Markov chain on $\{0, 1, \dots\}$ with transition probabilities given by

$$p_{01} = 1, \quad p_{ii+1} + p_{ii-1} = 1, \quad p_{ii+1} = \left(\frac{i+1}{i}\right)^2 p_{ii-1}, \quad i \geq 1\,.$$

Show that if $X_0 = 0$ the probability that $X_n \geq 1$ for all $n \geq 1$ is $6/\pi^2$.

13. In Example 8, which states are recurrent and which are transient?

14. A random sequence of non-negative integers $(F_n)_{n\geq 0}$ is obtained by setting $F_0 = 0$ and $F_1 = 1$ and, once $F_0, \dots, F_n$ are known, taking F_{n+1} to be either the sum or the difference of F_{n-1} and F_n, each with probability $1/2$. Is $(F_n)_{n\geq 0}$ a Markov chain? By considering the Markov chain $X_n = (F_{n-1}, F_n)$, find the probability that $(F_n)_{n\geq 0}$ reaches 3 before first returning to 0.

Draw enough of the flow diagram for $(X_n)_{n\geq 0}$ to establish a general pattern. Hence, using the strong Markov property, show that the hitting probability for $(1, 1)$, starting from $(1, 2)$, is $(3 - \sqrt{5})/2$. Deduce that $F_n \to \infty$ as $n \to \infty$.