STOCHASTIC CALCULUS AND APPLICATIONS

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Contents

Schedule

The course will develop stochastic calculus for continuous semimartingales and for jump processes. These techniques will be applied to the study of Markov processes in continuous time and space, through stochastic differential equations and in the identification of martingales. A range of applications will be discussed, both to the theory of random processes and to particular stochastic models. It is recommended that students have attended the Part III course Advanced Probability, or equivalent, or be prepared to refer to selected topics in the printed notes for that course.

Stochastic calculus: adaptedness and previsibility, finite variation processes, total variation, Lebesgue-Stieltjes integral, continuous local martingales, the Hilbert space $\mathcal{M}_c^2$ of continuous L^2-bounded martingales, quadratic variation of continuous local martingales, Itô integral, Itô formula, continuous semimartingales, Stratonovich integral, stochastic differential calculus.

Stochastic differential equations driven by Brownian motion: existence and uniqueness for Lipschitz coefficients, localization, examples, diffusion processes, relation with second order elliptic and parabolic partial differential equations.

Applications: identification of Brownian motions, exponential martingales, Girsanov's theorem, Cameron-Martin formula.

Markov jump processes Lévy kernel, stochastic integration with respect to an integer-valued random measure, stochastic calculus for jump processes, fluid limit.

Appropriate books

R. Durrett, *Probability: Theory and Examples.* Wadsworth 1991

O. Kallenberg, *Foundations of Morern Probability.* Springer 1997

B. Oksendal, *Stochastic Differential Equations: an introduction with applications.* Springer, 1992

D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion.* Springer, 1991.

L.C.G. Rogers and D. Williams, *Diffusions, Markov processes, and Martingales, Vol. 2: Itô calculus.* Wiley 1987

19. Stochastic calculus

19.1. **Adaptedness and previsibility.** Let $(\Omega, \mathcal{F})$ be a measurable space with filtration $(\mathcal{F}_t)_{t\geq 0}$. We will be concerned with two sorts of process. A *cadlag adapted process* X is a map $X : \Omega \times [0,\infty) \to \mathbb{R}$, such that $X_t = X(.,t)$ is $\mathcal{F}_t$-measurable for all $t \in [0,\infty)$, and $X(\omega,.) : [0,\infty) \to \mathbb{R}$ is cadlag for all $\omega \in \Omega$. We write $\Delta X_t = X_t - X_{t-}$ for $t > 0$. The *previsible σ-algebra* $\mathcal{P}$ on $\Omega \times (0,\infty)$ is the σ-algebra generated by sets of the form $A \times (s,t]$ with $A \in \mathcal{F}_s$ and $s \leq t$. A *previsible* process H is a $\mathcal{P}$-measurable map $H : \Omega \times (0,\infty) \to \mathbb{R}$.

Proposition 19.1.1. *Let X be a cadlag adapted process and set $H_t = X_{t-}$ for $t > 0$. Then H is previsible.*

Proposition 19.1.2. *Let H be a previsible process. Then H_t is $\mathcal{F}_t$-measurable for all $t \in (0,\infty)$.*

19.2. **Finite variation integrals.** Until further notice we take Ω to be a singleton $\{\omega\}$. Then a cadlag adapted process can be thought of as a cadlag map $X : [0,\infty) \to \mathbb{R}$, and a previsible process as a Borel measurable map $H : (0,\infty) \to \mathbb{R}$. So, whilst $\Omega = \{\omega\}$, we shall refer to cadlag processes and measurable processes respectively.

Let A be a cadlag process. For intervals $I = (s,t]$, set $dA((s,t]) = A_t - A_s$. Assume for now that A is increasing. Then dA extends uniquely to a Borel measure on $(0,\infty)$. (See Theorem 2.2.2.) Define, for a non-negative measurable process H and $t \geq 0$,

$$(H \cdot A)_t = \int_{(0,t]} H_s dA_s.$$

Proposition 19.2.1. *Let H be a non-negative measurable process and suppose that $(H \cdot A)_t < \infty$ for all t. Then $H \cdot A$ is a cadlag increasing process, with $\Delta(H \cdot A)_t = H_t \Delta A_t$ for all $t > 0$.*

Proposition 19.2.2. *Let H and K be non-negative measurable processes. Then*

$$H \cdot (K \cdot A) = (HK \cdot A).$$

Proposition 19.2.3. *Let H be non-negative, left continuous and locally bounded. Then*

$$(H \cdot A)_t = \lim_{n\to\infty} \sum_{k=0}^{\infty} H_{k2^{-n}} (A_{(k+1)2^{-n}\wedge t} - A_{k2^{-n}\wedge t}),$$

with convergence uniform on compact time intervals.

The assumption that A is increasing may be relaxed to an assumption that $A = A^+ - A^-$, where $A^\pm$ are increasing. Then, for any measurable process H, setting $H^\pm = \max(\pm H, 0)$, we define, subject to the finiteness of all the terms on the right,

$$H \cdot A = H^+ \cdot A^+ - H^+ \cdot A^- - H^- \cdot A^+ + H^- \cdot A^-.$$

Consistency can be checked by rearrangements. Propositions 19.2.1, 19.2.2 and 19.2.3 easily extend to this case.

We now investigate for which cadlag processes A there exist suitable increasing processes $A^{\pm}$. Write

$$\mathcal{D}_n = \{(k2^{-n}, (k+1)2^{-n}] : k \in \mathbb{Z}^+\}.$$

Set

$$V_t^n = \sum_{I \in \mathcal{D}_n, \inf I < t} |dA(I)|$$

and note that $V_t^n = U_t^n + |A_{t_n^+} - A_{t_n^-}|$, where

$$U_t^n = \sum_{I \in \mathcal{D}_n, \sup I < t} |dA(I)|$$

and $t_n^+ = 2^{-n}\lceil 2^n t \rceil, t_n^- = t_n^+ - 2^{-n}$. Now, as $n \to \infty$, U_t^n is increasing, so has a limit U_t, and $|A_{t_n^+} - A_{t_n^-}| \to |\Delta A_t|$. Hence V_t^n has a limit V_t and $V_t = U_t + |\Delta A_t|$. The process V is the *total variation process* of A. If V_t is finite for all t, then A has *finite variation.* Note that, when A is increasing, $V = A$, so A has finite variation. Assume that A has finite variation and set $A^{\pm} = \frac{1}{2}(V \pm A)$.

Proposition 19.2.4. *Let A be a cadlag finite variation process starting from* 0*, with total variation process V. Then*

(a) *V is cadlag, with $\Delta V_t = |\Delta A_t|$,*
(b) *$A = A^+ - A^-$ and A^+ and A^- are cadlag increasing processes,*
(c) *$A^{\pm}$ are the smallest processes satisfying* (b)*,*
(d) *$V = K \cdot A$ for some measurable process K with $|K| \equiv 1$,*
(e) *for all measurable processes H with $|H| \cdot V$ finite, $H \cdot A$ is a cadlag process with total variation process $|H| \cdot V$.*

We now drop the assumption that Ω is a singleton. For general Ω, the integral $H \cdot A$ and the total variation process V may be defined by fixing $\omega \in \Omega$ and referring to the special case previously considered.

Proposition 19.2.5. *Let A be a cadlag adapted finite variation process, with total variation process V. Then V is a cadlag adapted increasing process.*

Proposition 19.2.6. *Let A be a cadlag adapted finite variation process, with total variation process V, and let H and K be locally bounded previsible processes. Let T be a stopping time. Then*

(a) *$(H \cdot A)^T = (H1_{(0,T]} \cdot A) = (H \cdot A^T)$,*
(b) *$H \cdot A$ is a cadlag adapted finite variation process, with total variation process $|H| \cdot V$,*
(c) *$\Delta(H \cdot A)_t = H_t \Delta A_t$ for all $t > 0$,*
(d) *$H \cdot (K \cdot A) = (HK) \cdot A$.*

Proposition 19.2.7. *Let A be a cadlag adapted finite variation process and let H be a locally bounded, left continuous adapted process. Then*

$$(H \cdot A)_t = \lim_{n \to \infty} \sum_{k=0}^{\infty} H_{k2^{-n}} (A_{(k+1)2^{-n} \wedge t} - A_{k2^{-n} \wedge t})$$

with convergence uniform on compact time intervals, a.s.

19.3. **Martingales and local martingales.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $(\mathcal{F}_t)_{t \geq 0}$. Recall that an adapted integrable process X is a *martingale* if

$$\mathbb{E}(X_t | \mathcal{F}_s) = X_s \quad \text{a.s., for all } s \leq t.$$

Denote by $\mathcal{M}$ the set of cadlag martingales. (See Theorem 14.2.2 (path regularization) for context.)

By Theorem 14.3.6 (optional stopping), the set $\mathcal{M}$ is *stable under stopping*:

$$X \in \mathcal{M}, \quad T \text{ a stopping time} \quad \Rightarrow \quad X^T \in \mathcal{M}.$$

For any set of processes $\mathcal{C}$ that is stable under stopping, we write $\mathcal{C}_{loc}$ for the set of processes X such that, for some sequence of stopping times $T_n \uparrow \infty$ a.s., $X^{T_n} \in \mathcal{C}$ for all n. Note that $\mathcal{C}_{loc}$ inherits stability under stopping from $\mathcal{C}$. Often, a proposition true for $X \in \mathcal{C}$ can be generalized to $X \in \mathcal{C}_{loc}$ by considering the processes X^{T_n}. This is called a *localization argument.*

The elements of $\mathcal{M}_{loc}$ are called *local martingales.*

Proposition 19.3.1. *The following are equivalent:*

(a) *X is a martingale,*
(b) *X is a local martingale and, for all $t > 0$, the following set is UI:*

$$\{X_T : T \text{ a stopping time, } T \leq t\}.$$

Proposition 19.3.2. *Let M be a continuous local martingale starting from 0. Set $S_n = \inf\{t \geq 0 : |X_t| = n\}$. Then M^{S_n} is a martingale for all n.*

Proposition 19.3.3. *Let M be a continuous local martingale of finite variation starting from 0. Then $M \equiv 0$.*

This makes it clear that the theory developed in §19.2 is useless for continuous martingales.

Let $\mathcal{C}^2$ denote the set of cadlag processes X such that $|||X||| \equiv \|X^*\|_2 < \infty$, where $X^* = \sup_t |X_t|$. Let $\mathcal{M}^2$ denote the set of cadlag martingales which are bounded in L^2. (This coincides with the set $\mathcal{M}^2[0, \infty)$ from §14. The notation $\mathcal{M}^2$ was used also for a slightly different set in §12.) Write $\mathcal{M}^2_c$ for the subset of continuous martingales in $\mathcal{M}^2$. Recall from Theorem 14.3.4 that, for $X \in \mathcal{M}^2$, $X_t \to X_\infty$ as $t \to \infty$, a.s. and in L^2. For $X \in \mathcal{M}^2$, set $\|X\| = \|X_\infty\|_2$.

Proposition 19.3.4. *We have*

(a) $(\mathcal{C}^2, |||\dots|||)$ *is complete,*
(b) $\mathcal{M}^2 = \mathcal{M} \cap \mathcal{C}^2$,
(c) $(\mathcal{M}^2, \|\dots\|)$ *and* $(\mathcal{M}^2_c, \|\dots\|)$ *are Hilbert spaces,*
(d) $X \mapsto X_\infty : \mathcal{M}^2 \to L^2(\mathcal{F}_\infty)$ *is an isometry.*

19.4. **Itô integrals I.** Let S and T be stopping times with $S \leq T$ and let Z be a bounded $\mathcal{F}_S$-measurable random variable. Set

$$H_t = \begin{cases} Z & \text{if } S < t \leq T \\ 0 & \text{otherwise.} \end{cases}$$

We write $H = Z1_{(S,T]}$. Such processes H are called *elementary* and the set of elementary processes is denoted by $\mathcal{E}$.

Given a sequence of stopping times $0 = T_0 \leq T_1 \leq \dots \leq T_k \to \infty$ a.s. and uniformly bounded $\mathcal{F}_{T_k}$-measurable random variables Z_k, set

$$H = \sum_{k=0}^{\infty} Z_k 1_{(T_k, T_{k+1}]}.$$

Such processes are called *simple* and the set of simple processes is denoted $\mathcal{S}$.

For any reasonable definition, the integral $(1_{(S,T]} \cdot M)_t$ must be given by

$$M_{T\wedge t} - M_{S\wedge t} = \begin{cases} 0 & \text{if } t < S, \\ M_t - M_S & \text{if } S \leq t < T, \\ M_T - M_S & \text{if } T \leq t. \end{cases}$$

If we impose linearity, this dictates the form of the integral $H \cdot M$ for $H \in \mathcal{S}$:

$$(H \cdot M)_t = \sum_{k=0}^{\infty} Z_k (M_{T_{k+1}\wedge t} - M_{T_k \wedge t}).$$

Note that, as $T_k \to \infty$ a.s., the sum is a.s. finite for each t.

Proposition 19.4.1. *Let* $H \in \mathcal{S}$ *and* $M \in \mathcal{M}^2$*. Let* T *be a stopping time. Then*
(a) $(H \cdot M)^T = (H1_{(0,T]} \cdot M) = (H \cdot M^T)$,
(b) $H \cdot M \in \mathcal{M}^2$,
(c)

$$\mathbb{E}[(H \cdot M)^2_\infty] = \sum_{k=0}^{\infty} \mathbb{E}[Z_k^2 (M^2_{T_{k+1}} - M^2_{T_k})] \leq \|H\|^2_\infty \|M\|^2.$$

19.5. **Quadratic variation.** Say that a sequence of processes X^n converges to X *uniformly on compacts in probability* or *u.c.p.* if, for all $\varepsilon > 0$ and $t \geq 0$,

$$\mathbb{P}\left(\sup_{s\leq t} |X^n_s - X_s| > \varepsilon\right) \to 0.$$

Theorem 19.5.1. (a) *For $M \in \mathcal{M}_{c,loc}$, the following limit exists u.c.p.:*

$$[M]_t = \lim_{n\to\infty} \sum_{k=0}^{[2^n t]-1} (M_{(k+1)2^{-n}} - M_{k2^{-n}})^2$$

and defines a continuous, increasing adapted process $[M]$.

(b) *Moreover $[M]$ is the unique increasing process A starting from 0 such that $M^2 - A \in \mathcal{M}_{c,loc}$.*

(c) *For $M \in \mathcal{M}_c^2$, $M^2 - [M]$ is a UI martingale.*

The process $[M]$ is called the *quadratic variation* of M.

19.6. **Itô integrals II.** Given $M \in \mathcal{M}_c^2$, there is a unique measure μ on $\mathcal{P}$ such that

$$\mu(A \times (s,t]) = \mathbb{E}(1_A([M]_t - [M]_s)), \quad A \in \mathcal{F}_s, s \le t.$$

Set $L^2(M) = L^2(\mu, \mathcal{P})$. Then a previsible process $H \in L^2(M)$ if and only if

$$\|H\|_M^2 \equiv \mathbb{E}[(H^2 \cdot [M])_\infty] < \infty.$$

Given $0 = t_0 \le t_1 \le \cdots \le t_n$ and uniformly bounded $\mathcal{F}_{t_k}$-measurable random variables Z_k, set

$$H = \sum_{k=0}^{n-1} Z_k 1_{(t_k, t_{k+1}]}.$$

Such processes are called *strictly simple* and the set of strictly simple processes is denoted $\mathcal{S}^*$.

Proposition 19.6.1. *$\mathcal{S}^*$ is dense in $L^2(M)$.*

Proposition 19.6.2. *For $M \in \mathcal{M}_c^2$ and $H \in \mathcal{S}^*$ set $I(H) = H \cdot M$. Then $I : (\mathcal{S}^*, \|\dots\|_M) \to (\mathcal{M}_c^2, \|\dots\|)$ is an isometry.*

Since $(\mathcal{M}_c^2, \|\dots\|)$ is a Hilbert space, we deduce:

Theorem 19.6.3. *There exists a unique isometry $I : L^2(M) \to \mathcal{M}_c^2$ such that $I(H) = H \cdot M$ for $H \in \mathcal{S}^*$.*

We write

$$I(H)_t = (H \cdot M)_t = \int_0^t H_s dM_s.$$

The process $(H \cdot M)$ is called the *Itô integral* of H with respect to M. It depends for its existence, not on M having finite variation, but on the fact that M is a martingale, so it is a *stochastic integral.*

Proposition 19.6.4. *Let $H \in L^2(M)$ and $M \in \mathcal{M}_c^2$. Let T be a stopping time. Then*

$$(H \cdot M)^T = (H1_{(0,T]} \cdot M) = (H \cdot M^T).$$

Say that a process H is *locally bounded* if there is a sequence of stopping times $T_n \uparrow \infty$ a.s. such that $H1_{(0,T_n]}$ is uniformly bounded for all n. For H locally bounded previsible and $M \in \mathcal{M}_{c,loc}$, there exist stopping times $T_n \uparrow \infty$ a.s. such that $H1_{(0,T_n]}$ is uniformly bounded previsible and $M^{T_n} \in \mathcal{M}_c^2$ for all n. Set

$$(H \cdot M)_t = ((H1_{(0,T_n]}) \cdot M^{T_n})_t \quad \text{for } t \leq T_n.$$

The obvious consistency questions are settled by the preceding proposition. Thus we obtain an extension of the Itô integral.

Proposition 19.6.5. *Let $M \in \mathcal{M}_{c,loc}$ and let H and K be locally bounded previsible process. Let T be a stopping time. Then*

(a) $(H \cdot M)^T = (H1_{(0,T]} \cdot M) = (H \cdot M^T)$,
(b) $H \cdot M \in \mathcal{M}_{c,loc}$,
(c) $[H \cdot M] = H^2 \cdot [M]$,
(d) $H \cdot (K \cdot M) = (HK) \cdot M$.

Proposition 19.6.6. *Let $M \in \mathcal{M}_{c,loc}$ and let H be a locally bounded left-continuous adapted process. Then*

$$\sum_{k=0}^{[2^n t]-1} H_{k2^{-n}}(M_{(k+1)2^{-n}} - M_{k2^{-n}}) \to (H \cdot M)_t \quad u.c.p.$$

Proposition 19.6.7. (a) *For $M, N \in \mathcal{M}_{c,loc}$, the following limit exists u.c.p.:*

$$[M, N]_t = \lim_{n \to \infty} \sum_{k=0}^{[2^n t]-1} (M_{(k+1)2^{-n}} - M_{k2^{-n}})(N_{(k+1)2^{-n}} - N_{k2^{-n}})$$

and defines a continuous, finite variation, adapted process $[M, N]$.

(b) *Moreover $[M, N]$ is the unique finite variation process A starting from 0 such that $MN - A \in \mathcal{M}_{c,loc}$.*
(c) *For $M, N \in \mathcal{M}_c^2$, $MN - [M, N]$ is a UI martingale.*
(d) *For $M, N \in \mathcal{M}_{c,loc}$ and locally bounded previsible processes H, K, we have*

$$[H \cdot M, K \cdot N] + [K \cdot M, H \cdot N] = 2(HK) \cdot [M, N].$$

The process $[M, N]$ is called the *covariation* of M and N.

Proposition 19.6.8 (Kunita–Watanabe identity). *Let $M, N \in \mathcal{M}_{c,loc}$ and let H be a locally bounded previsible process. Then*

$$[H \cdot M, N] = H \cdot [M, N].$$

19.7. **Itô calculus.** A *continuous semimartingale* is any process X having a *Doob–Meyer* decomposition:

$$X_t = X_0 + M_t + A_t$$

where X_0 is $\mathcal{F}_0$-measurable, $M_0 = A_0 = 0$, $M \in \mathcal{M}_{c,loc}$ and A is continuous, adapted and of finite variation. Proposition 19.3.3 shows that any such decomposition is unique.

Set $[X] = [M]$ and write $[X, Y]$ for the corresponding bilinear form for semimartingales X, Y. Then

$$[X]_t = \lim_{n\to\infty} \sum_{k=0}^{[2^n t]-1} (X_{(k+1)2^{-n}} - X_{k2^{-n}})^2 \quad \text{u.c.p.}$$

Thus *quadratic variation does not see the finite variation part.*

For H locally bounded previsible, set

$$H \cdot X = H \cdot M + H \cdot A$$

or, in alternative notation,

$$\int_0^t H_s dX_s = \int_0^t H_s dM_s + \int_0^t H_s dA_s,$$

where the first integral on the right is the Itô integral and the second is the Lebesgue–Stieltjes integral.

Proposition 19.7.1 (Integration by parts)**.** *Let X and Y be continuous semimartingales. Then*

$$X_t Y_t - X_0 Y_0 = \int_0^t X_s dY_s + \int_0^t Y_s dX_s + [X, Y]_t.$$

Theorem 19.7.2 (Itô formula)**.** *Let $X^1, \dots, X^n$ be continuous semimartingales and set $X = (X^1, \dots, X^n)$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be C^2. Then*

$$f(X_t) - f(X_0) = \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x^i}(X_s) dX_s^i + \tfrac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s) d[X^i, X^j]_s.$$

19.8. **Stratonovich integral.** For continuous semimartingales X and Y, define the *Stratonovich integral*

$$\int_0^t Y_s \partial X_s = \int_0^t Y_s dX_s + \tfrac{1}{2}[X, Y]_t.$$

From the discrete approximations to the terms on the right, we obtain u.c.p.

$$\int_0^t Y_s \partial X_s = \lim_{n\to\infty} \sum_{k=0}^{[2^n t]-1} \left(\frac{Y_{(k+1)2^{-n}} + Y_{k2^{-n}}}{2} \right) (X_{(k+1)2^{-n}} - X_{k2^{-n}}).$$

Note that the integrand Y is evaluated symmetrically with respect to the interval over which the increment of the integrator X is taken, rather than at the left endpoint in the Itô integral.

The integration by parts formula can be rewritten (trivially) as

$$X_tY_t - X_0Y_0 = \int_0^t X_s\partial Y_s + \int_0^t Y_s\partial X_s.$$

Proposition 19.8.1 (Chain rule). *Let $X^1,\dots,X^n$ be continuous semimartingales and set $X=(X^1,\dots,X^n)$. Let $f:\mathbb{R}^n\to\mathbb{R}$ be C^3. Then*

$$f(X_t) - f(X_0) = \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x^i}(X_s)\partial X_s^i.$$

19.9. **Stochastic differential calculus.** The main properties of Itô and Stratonovich calculus can be encoded in an intuitively plausible calculus of differentials. Suppose we agree that

$$\begin{aligned} dZ_t = H_t dX_t &\quad \text{means} \quad Z_t - Z_0 = \int_0^t H_s dX_s, \\ \partial Z_t = Y_t\partial X_t &\quad \text{means} \quad Z_t - Z_0 = \int_0^t Y_s\partial X_s, \\ dZ_t = dX_t dY_t &\quad \text{means} \quad Z_t - Z_0 = [X,Y]_t \end{aligned}$$

and so does $\partial Z_t = \partial X_t\partial Y_t$. Then we can express Proposition 19.6.5(d), the Kunita–Watanabe formula, integration by parts formula and Itô formula as follows:

$$\begin{aligned} H_t(K_t dX_t) &= (H_tK_t)dX_t \\ H_t(dX_t dY_t) &= (H_t dX_t)dY_t \\ d(X_tY_t) &= (dX_t)Y_t + X_t(dY_t) + dX_t dY_t \\ d(f(X_t)) &= \frac{\partial f}{\partial x^i}(X_t)dX_t^i + \tfrac{1}{2}\frac{\partial^2 f}{\partial x^i\partial x^j}(X_t)dX_t^i dX_t^j \end{aligned}$$

or, in tensor notation,

$$d(f(X_t)) = Df(X_t)dX_t + \tfrac{1}{2}D^2f(X_t)dX_t\otimes dX_t.$$

The relationship between Itô and Stratonovich integrals is expressed by

$$\partial Z_t = Y_t\partial X_t \quad \text{if and only if} \quad dZ_t = Y_t dX_t + \tfrac{1}{2}dX_t dY_t.$$

Moreover

$$\begin{aligned} X_t(Y_t\partial Z_t) &= (X_tY_t)\partial Z_t \\ X_t(\partial Y_t\partial Z_t) &= (X_t\partial Y_t)\partial Z_t \\ \partial(X_tY_t) &= (\partial X_t)Y_t + X_t(\partial Y_t) \\ \partial(f(X_t)) &= Df(X_t)\partial X_t. \end{aligned}$$

19.10. **Locally defined processes.** A *locally defined process* (X,ζ) consists of a stopping time ζ together with a measurable map

$$X : \{(\omega,t) \in \Omega \times [0,\infty) : t < \zeta(\omega)\} \to \mathbb{R}.$$

It is *cadlag* (respectively *continuous*) if the map

$$t \mapsto X_t(\omega) : [0,\zeta(\omega)) \to \mathbb{R}$$

is cadlag (respectively continuous) for all ω. It is *adapted* if

$$X_t : \Omega_t \equiv \{\omega : t < \zeta(\omega)\} \to \mathbb{R}$$

is $\mathcal{F}_t$-measurable for all t. We say that (X,ζ) is a *locally defined local martingale* if there exist stopping times $T_n \uparrow \zeta$ a.s. and martingales X^n such that

$$X_t = X_t^n, \quad \text{for all } t \le T_n, \text{ for all } n.$$

Proposition 19.10.1 (Local Itô formula). *Let $X^1,\dots,X^n$ be continuous semimartingales and set $X = (X^1,\dots,X^n)$. Let $U \subseteq \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}$ be C^2. Set*

$$\zeta = \inf\{t \ge 0 : X_t \notin U\}.$$

Then, for all $t < \zeta$,

$$f(X_t) - f(X_0) = \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x^i}(X_s)dX_s^i + \tfrac{1}{2}\sum_{i,j=1}^n \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s)d[X^i,X^j]_s.$$

20. Stochastic differential equations

20.1. **Existence and uniqueness.** Recall, for $U \subseteq \mathbb{R}^d$, that a function $f : U \to \mathbb{R}$ is *Lipschitz (on U)* if, for some $K < \infty$, for all $x, y \in U$, we have $|f(x)-f(y)| \le K|x-y|$. For U open, a function $f : U \to \mathbb{R}$ is *locally Lipschitz* if f is Lipschitz on every compact subset of U.

We suppose given an open set U in $\mathbb{R}^d$, a starting point $x_0 \in U$, locally Lipschitz *coefficients*

$$\sigma : U \to \mathbb{R}^d \otimes (\mathbb{R}^m)^*, \qquad b : U \to \mathbb{R}^d,$$

and a Brownian motion $B = (B^1,\dots,B^m)$ in $\mathbb{R}^m$.

Consider the stochastic differential equation

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt. \tag{20.1}$$

By a *solution to (20.1), starting from x_0*, we mean a continuous adapted process X, with values in U, such that, for all $t \ge 0$,

$$X_t = x_0 + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds. \tag{20.2}$$

More generally, a *local solution to (20.1), starting from* x_0 is a continuous adapted locally defined process (X, ζ) with values in U such that (20.2) holds for all $t < \zeta$.

Denote by $\mathcal{C}$ the set of continuous adapted processes X such that, for all $T > 0$,

$$|||X|||_T \equiv \| \sup_{t \le T} |X_t| \|_2 < \infty.$$

Denote by $\mathcal{C}_T$ the set of continuous adapted processes X indexed by $[0, T]$ such that $|||X|||_T < \infty$. Note that $(\mathcal{C}_T, |||\dots|||_T)$ is complete.

Theorem 20.1.1. *Assume that* $U = \mathbb{R}^d$ *and that* σ *and* b *are Lipschitz. Then there exists a unique* $X \in \mathcal{C}$ *such that* X *is a solution to (20.1) starting from* x_0.

Theorem 20.1.2. *Assume that* U *is an open set in* $\mathbb{R}^d$, *and that* σ *and* b *are locally Lipschitz. Then there exists a unique maximal local solution* (X, ζ) *to (20.1) starting from* x_0. *Moreover, for all compacts* $C \subseteq U$, *on the set* $\{\zeta < \infty\}$, *we have*

$$\sup\{t < \zeta : X_t \in C\} < \zeta, \quad a.s.$$

From now on, unless an open set U is mentioned explicitly, we will take $U = \mathbb{R}^d$.

20.2. **Uniqueness in law.** Let (X, ζ) be the unique maximal local solution to (20.1) starting from x_0 provided by Theorem 20.1.2. Let $W = C([0, \infty), \mathbb{R}^m)$ and let $\mathcal{W}$ and $(\mathcal{W}_t)_{t \ge 0}$ be, respectively, the σ-algebra and the filtration on W generated by the coordinate process β. Let μ denote Wiener measure on $\mathcal{W}$. By Theorem 20.1.2, the stochastic differential equation

$$dY_t = \sigma(Y_t) d\beta_t + b(Y_t) dt$$

has a unique maximal local solution (Y, η) starting from x_0.

Proposition 20.2.1. *We have* $(X, \zeta) = (Y \circ B, \eta \circ B)$ *a.s. In particular, the laws of* (X, ζ) *under* $\mathbb{P}$ *and* (Y, η) *under* μ *coincide.*

20.3. **Diffusion processes.** Let there be given bounded measurable functions

$$a : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d, \qquad b : \mathbb{R}^d \to \mathbb{R}^d$$

where a is symmetric. Set, for $f \in C^2(\mathbb{R}^d)$,

$$Lf = \tfrac{1}{2} \sum_{i,j=1}^d a^{ij}(x) \frac{\partial^2 f}{\partial x^i \partial x^j} + \sum_{i=1}^d b^i(x) \frac{\partial f}{\partial x^i}.$$

Let X be a continuous adapted process. We say X is an *L-diffusion* if, for all $f \in C_b^2(\mathbb{R}^d)$, the following process is a martingale:

$$M_t^f = f(X_t) - f(X_0) - \int_0^t Lf(X_s) ds.$$

Alternatively we say that X is a *diffusion* with *diffusivity* a and *drift* b.

Proposition 20.3.1. *Suppose X is a solution to (20.1). Set $a = \sigma\sigma^*$. Then X is an L-diffusion. Indeed, for all $f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$, the following process is a martingale:*

$$M_t^f = f(t, X_t) - f(0, X_0) - \int_0^t (\partial/\partial s + L) f(s, X_s) ds.$$

Note that if a and b are Lipschitz, and if a satisfies, for some $\varepsilon > 0$,

$$\langle \xi, a(x)\xi \rangle \geq \varepsilon |\xi|^2, \qquad \xi \in (\mathbb{R}^d)^*. \tag{20.3}$$

then we can find a Lipschitz function $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes (\mathbb{R}^d)^*$ such that $a = \sigma\sigma^*$. (A canonical way to do this is to identify $\mathbb{R}^d$ and $(\mathbb{R}^d)^*$ using the standard basis, then take σ to be the positive-definite square root of a, which exists and is Lipschitz by (20.3).) Hence, under these conditions, from existence of solutions to (20.1), we can deduce the existence of L-diffusions.

Proposition 20.3.2. *Let X be an L-diffusion. Then, for all $f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$, the following process is a martingale:*

$$M_t^f = f(t, X_t) - f(t, X_0) - \int_0^t (\partial/\partial s + L) f(s, X_s) ds.$$

Proposition 20.3.3. *Let X be an L-diffusion and let T be a finite stopping time. Set $\tilde{X}_t = X_{T+t}$ and $\tilde{\mathcal{F}}_t = \mathcal{F}_{T+t}$. Then $\tilde{X}$ is an L-diffusion with respect to $(\tilde{\mathcal{F}}_t)_{t \geq 0}$.*

We will see in Theorem 20.4.5 that, when a and b are Lipschitz and a satisfies (20.3), the L-diffusion property implies that, conditional on X_0, X is independent of $\mathcal{F}_0$ with a distribution uniquely determined by L. Then Proposition 20.3.3 becomes a strong Markov property.

20.4. **Relation with second order elliptic and parabolic partial differential equations.** Let a and b be Lipschitz and let a satisfy (20.3). Let D be a bounded open set in $\mathbb{R}^d$ with smooth boundary ∂D and denote by λ the surface area measure on ∂D.

Theorem 20.4.1 (Dirichlet problem I). *For each $f \in C(\partial D)$, there exists a unique $u \in C(\bar{D}) \cap C^2(D)$ such that*

$$\begin{cases} Lu = 0 & \text{on } D, \\ u = f & \text{on } \partial D. \end{cases}$$

Moreover, there is a continuous function $m : D \times \partial D \to (0, \infty)$ such that, for all $f \in C(\partial D)$, u is given by

$$u(x) = \int_{\partial D} m(x, y) f(y) \lambda(dy).$$

We call $m(x, y)\lambda(dy)$ the *harmonic measure on ∂D starting from x.*

Theorem 20.4.2 (Dirichlet problem II). *For each $\phi \in L^\infty(D)$, there exists a unique $u \in C(\bar{D}) \cap C^2(D)$ such that*

$$\begin{cases} Lu + \phi = 0 & \text{on } D, \\ u = 0 & \text{on } \partial D. \end{cases}$$

Moreover, there is a continuous function $m : \{(x,y) \in D \times D : x \neq y\} \to (0,\infty)$ such that, for all $\phi \in L^\infty(D)$, u is given by

$$u(x) = \int_D g(x,y)\phi(y)dy.$$

We call $g(x,y)$ the *Green kernel.*

Theorem 20.4.3. *Assume that $f \in C(\partial D)$ and $\phi \in L^\infty(D)$. Let $u \in C(\bar{D}) \cap C_b^2(D)$ satisfy*

$$\begin{cases} Lu + \phi = 0 & \text{on } D, \\ u = f & \text{on } \partial D. \end{cases}$$

Then, for any L-diffusion X, for all $x \in D$,

$$u(x) = \mathbb{E}_x\left[\int_0^T \phi(X_s)ds + f(X_T)\right],$$

where $T = \inf\{t \geq 0 : X_t \notin D\}$. In particular, for Borel sets $A \subseteq D$ and $B \subseteq \partial D$,

$$\mathbb{E}_x \int_0^T 1_{X_s \in A} ds = \int_A g(x,y)dy,$$

$$\mathbb{P}_x(X_T \in B) = \int_B m(x,y)\lambda(dy).$$

Theorem 20.4.4 (Cauchy problem). *For each $f \in C_b^2(\mathbb{R}^d)$, there exists a unique $u \in C_b^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$ such that*

$$\begin{cases} \partial u/\partial t = Lu & \text{on } \mathbb{R}^+ \times \mathbb{R}^d, \\ u(0,.) = f & \text{on } \mathbb{R}^d. \end{cases}$$

Moreover, there is a continuous function $p : (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to (0,\infty)$ such that, for all $f \in C_b^2(\mathbb{R}^d)$, u is given by

$$u(t,x) = \int_{\mathbb{R}^d} p(t,x,y)f(y)dy.$$

We call $p(t,x,y)$ the *heat kernel.*

Theorem 20.4.5. *Assume that $f \in C_b^2(\mathbb{R}^d)$. Let $u \in C_b^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$ satisfy*

$$\begin{cases} \partial u/\partial t = Lu & \text{on } \mathbb{R}^+ \times \mathbb{R}^d, \\ u(0,.) = f & \text{on } \mathbb{R}^d. \end{cases}$$

Then, for any L-diffusion X, for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^d$,

$$u(t,x) = \mathbb{E}_x(f(X_t)).$$

In particular, for Borel sets $A \subseteq \mathbb{R}^d$,

$$\mathbb{P}_x(X_t \in A) = \int_A p(t,x,y)dy.$$

Moreover, for all $s \leq t$,

$$\mathbb{E}_x(f(X_t)|\mathcal{F}_s) = u(t-s, X_s)$$

so, conditional on X_0, X *is independent of* $\mathcal{F}_0$ *with a distribution uniquely determined by* L.

Theorem 20.4.6 (Feynman–Kac formula)**.** *Assume that* $f \in C_b^2(\mathbb{R}^d)$ *and that* $V \in L^\infty(\mathbb{R}^d)$. *Let* $u \in C_b^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$ *satisfy*

$$\begin{cases} \partial u/\partial t = \frac{1}{2}\Delta u + V.u & \text{on } \mathbb{R}^+ \times \mathbb{R}^d, \\ u(0,.) = f & \text{on } \mathbb{R}^d. \end{cases}$$

Then, for all $t \in \mathbb{R}^+$ *and* $x \in \mathbb{R}^d$,

$$u(t,x) = \mathbb{E}_x\left(f(B_t)\exp\left\{\int_0^t V(B_s)ds\right\}\right).$$

21. Applications

21.1. Identification of Brownian motions.

Theorem 21.1.1 (Lévy's characterization of Brownian motion)**.** *Let* $M^1, \ldots, M^n$ *be continuous local martingales starting from* 0, *with* $[M^i, M^j]_t = \delta_{ij}t$. *Then* $M = (M^1, \ldots, M^n)$ *is a Brownian motion.*

Proposition 21.1.2. *Let* B *be a Brownian motion and let* $h \in L^2(\mathbb{R}^+)$. *Set*

$$X = \int_0^\infty h_s dB_s.$$

Then $X \sim N(0, \|h\|_2^2)$.

Theorem 21.1.3. *Let* M *be a continuous local martingale starting from* 0, *with* $[M]_\infty = \infty$ *a.s. Set*

$$\tau_s = \inf\{t > 0 : [M]_t > s\}, \quad \tilde{B}_s = M(\tau_s), \quad \tilde{\mathcal{F}}_s = \mathcal{F}(\tau_s).$$

Then τ_s *is a stopping time and* $[M](\tau_s) = s$ *for all* $s \geq 0$. *Also* $M_t = \tilde{B}([M]_t)$ *for all* $t \geq 0$. *Moreover* $\tilde{B}$ *is an* $(\tilde{\mathcal{F}}_t)_{t\geq 0}$*-Brownian motion.*

Example (Ornstein–Uhlenbeck process)**.** Consider the SDE in $\mathbb{R}^2$

$$\begin{aligned} dX_t =& dB_t - \lambda X_t dt, \quad X_0 = x_0, \\ dY_t =& X_t dt, \quad Y_0 = y_0. \end{aligned}$$

For $\lambda > 0$, Y can be used as a model of random particle motion. Note that X then describes the velocity of Y. We call X the *Ornstein–Uhlenbeck (velocity) process.*

This is a rare example of an SDE with an explicit solution. Compute by Itô's formula

$$d(e^{\lambda t}X_t) = \lambda e^{\lambda t}X_t dt + e^{\lambda t}dX_t = e^{\lambda t}dB_t.$$

Hence

$$X_t = e^{-\lambda t}\{x_0 + \int_0^t e^{\lambda s}dB_s\},$$

so, by Proposition 21.1.2, $X \sim N(x_0 e^{-\lambda t}, (1 - e^{-2\lambda t})/2\lambda)$.

Example (Bessel processes). Consider for $\nu \in [1, \infty)$ the SDE

$$dX_t = dB_t + \left(\frac{\nu - 1}{2X_t}\right) dt$$

in the open set $U = (0, \infty)$. We know that, for all $x_0 \in U$, there is a unique maximal local solution (X, ζ) starting from x_0, with

$$\zeta = \lim_{k\to\infty}(\inf\{t \geq 0 : X_t \leq 1/k\}).$$

We call (X, ζ) a *Bessel process* of *dimension* ν.

Consider the case $\nu \in \mathbb{N}$. Let B be a Brownian motion in $\mathbb{R}^\nu$ and set $Y_t = |B_t|$. By the local Itô formula, for $t < \eta \equiv \inf\{t \geq 0 : B_t = 0\}$,

$$dY_t = \frac{\langle B_t, dB_t\rangle}{|B_t|} + \left(\frac{\nu - 1}{2|B_t|}\right) dt.$$

Set (for all $t \geq 0$)

$$\beta_t = \int_0^t \frac{\langle B_s, dB_s\rangle}{|B_s|}$$

then β is a continuous local martingale and

$$d\beta_t d\beta_t = |B_t|^{-2} \sum_{i,j=1}^{\nu} B_t^i B_t^j dB_t^i dB_t^j = dt$$

so β is a Brownian motion by Lévy's characterization. We have

$$dY_t = d\beta_t + \left(\frac{\nu - 1}{2Y_t}\right) dt$$

so $(Y, \eta) \sim (X, \zeta)$ by uniqueness in law. We know, for $\nu \geq 2$, that $\eta = \infty$ a.s., so also $\zeta = \infty$ a.s.

21.2. Exponential martingales.

Example. Let M be a continuous local martingale starting from 0. Set $Z_t = \exp(M_t - \frac{1}{2}[M]_t)$. We write $Z = \mathcal{E}(M)$ and call Z the *stochastic exponential* of M. Then, by Itô's formula,

$$dZ_t = Z_t(dM_t - \tfrac{1}{2}d[M]_t) + \tfrac{1}{2}Z_t(dM_t - \tfrac{1}{2}d[M]_t)(dM_t - \tfrac{1}{2}d[M]_t) = Z_t dM_t.$$

So Z is also a continuous local martingale.

Proposition 21.2.1 (Exponential martingale inequality)**.** *Let M be a continuous local martingale starting from 0. Then, for all $\varepsilon, \delta > 0$,*

$$\mathbb{P}\left(\sup_{t\geq 0} M_t \geq \varepsilon \quad \text{and} \quad [M]_\infty \leq \delta\right) \leq e^{-\varepsilon^2/2\delta}.$$

Proposition 21.2.2. *Let M be a continuous local martingale, starting from 0, and suppose that $[M]$ is uniformly bounded. Then $\mathcal{E}(M)$ is a uniformly integrable martingale.*

Theorem 21.2.3 (Girsanov's theorem)**.** *Let M be a continuous local martingale, starting from 0, and suppose that $Z \equiv \mathcal{E}(M)$ is a uniformly integrable martingale. Then we can define a new probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ by*

$$\tilde{\mathbb{P}}(A) = \mathbb{E}(Z_\infty 1_A), \quad A \in \mathcal{F}.$$

Moreover, if $X \in \mathcal{M}_{c,loc}(\mathbb{P})$ then $X - [X, M] \in \mathcal{M}_{c,loc}(\tilde{\mathbb{P}})$.

Let $(W, \mathcal{W}, \mu)$ be Wiener space. (See §12.4.3.) Thus $W = C(\mathbb{R}^+, \mathbb{R})$, $\mathcal{W}$ is the σ-algebra generated by the coordinate functions and μ is Wiener measure, the law of Brownian motion. Define the *Cameron–Martin space*

$$H = \left\{ h \in W : h_t = \int_0^t \phi_s ds \quad \text{for some} \quad \phi \in L^2(\mathbb{R}^+) \right\}.$$

For $h \in H$, we write $\dot{h}$ for a version of the weak derivative ϕ.

Theorem 21.2.4 (Cameron–Martin theorem)**.** *Fix $h \in H$ and define the shifted measure*

$$\mu^h(A) = \mu(\{w \in W : w + h \in A\}), \quad A \in \mathcal{W}.$$

Then μ^h is absolutely continuous with respect to μ, with Radon–Nikodym derivative given by

$$\frac{d\mu^h}{d\mu}(w) = \exp\left\{ \int_0^\infty \dot{h}_s dw_s - \tfrac{1}{2}\int_0^\infty |\dot{h}_s|^2 ds \right\}, \quad \mu\text{-a.a.}w.$$

22. Markov jump processes

22.1. **Construction and basic properties.** Let I be a measurable subset in $\mathbb{R}^d$. Let K be a *kernel* on I, that is, a map

$$K : I \times \mathcal{B}(\mathbb{R}^d) \to [0, \infty]$$

such that

(a) $K(., A)$ is measurable for all A,
(b) $K(x, .)$ is a measure for all x.

We suppose also that K is *bounded*

(c) $\lambda(x) \equiv K(x, \mathbb{R}^d) \leq C$ for all x, for some $C < \infty$

and

(d) $x + y \in I$ for $K(x, .)$-almost all y, for all x.

Such a kernel K gives the basic data for a pure jump Markov process X in I.

Two pieces of information are encoded in K: the function $\lambda(x) = K(x, \mathbb{R}^d)$ gives the rate of jumping from x, and the probability kernel $P(x, .) = K(x, .)/\lambda(x)$ gives the distribution of the jump from x.

From P alone, given an initial distribution π on I, we can specify the law of a discrete-time process $(Z_n)_{n \geq 0}$ by the requirements $Z_0 \sim \pi$ and

$$\mathbb{E}(f(Z_{n+1})|\mathcal{Z}_n) = \int_{\mathbb{R}^d} f(Z_n + y) P(Z_n, dy),$$

where $\mathcal{Z}_n = \sigma(Z_0, \ldots, Z_n)$. We say that Z is a *discrete-time Markov process with transition kernel* P, or just that Z is *Markov*(P). Note that condition (d) above ensures that Z remains in I.

In the case where I is countable, we can construct a Markov(P) process from a countable collection of independent random variables as follows: take $Z_0 \sim \pi$ and, for each $x \in I$ and $n \in \mathbb{N}$, take $Y_n(x) \sim P(x, .)$. Set

$$Z_n = Z_0 + Y_1(Z_0) + \cdots + Y_n(Z_{n-1}).$$

Then Z is Markov(P).

Given any Markov(P) process Z, we can construct a continuous-time process X by *Poissonization*. In the simplest case, where λ is a constant independent of x, we just set $X_t = Z_{N(t)}$, where N is a Poisson process of rate λ. Generally, we take a Poisson process N of rate 1 and set $X_t = Z_{N(A_t)}$, where

$$A_t = \int_0^t \lambda(X_s) ds.$$

This definition is made for each ω. It works progressively, jump by jump, and is not circular.

The *generator* L associated with K, mapping bounded measurable functions to bounded measurable functions, is given by

$$Lf(x) = \int_{\mathbb{R}^d} \{f(x+y) - f(x)\} K(x, dy).$$

The process X is a *continuous-time Markov process* with *generator* L, equivalently, with *Lévy kernel* K. We write also $X \sim$*Markov*(L).

Given a filtration $(\mathcal{F}_t)_{t\geq 0}$, we say that a function

$$H : \Omega \times (0, \infty) \times \mathbb{R}^d \to \mathbb{R}$$

is *previsible* if it is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, where $\mathcal{P}$ is the previsible σ-algebra on $\Omega \times (0, \infty)$.

A cadlag process X is said to be *pure jump* if it has only finitely many jumps in any finite time interval and is constant between jumps. Given a pure jump process X, write J_n and Y_n for the time and size, respectively, of the nth jump. The *jump measure* of X is the random integer-valued measure μ on $(0, \infty) \times \mathbb{R}^d$ defined by

$$\mu = \sum_{n=1}^{\infty} \varepsilon_{(J_n, Y_n)}.$$

Note that, for any measurable function f,

$$f(X_t) = f(X_0) + \int_0^t \int_{\mathbb{R}^d} \{f(X_{s-} + y) - f(X_{s-})\} \mu(ds, dy).$$

Theorem 22.1.1. *Let X be a cadlag process with natural filtration $(\mathcal{F}_t)_{t\geq 0}$. The following are equivalent:*

(a) *X is Markov(L),*

(b) *X is a pure jump process and, for all previsible processes H such that*

$$\mathbb{E} \int_0^t \int_{\mathbb{R}^d} |H(s, y)| \nu(ds, dy) < \infty,$$

the following process is a martingale:

$$M_t^H = \int_0^t \int_{\mathbb{R}^d} H(s, y)(\mu - \nu)(ds, dy)$$

where μ is the jump measure of X and $\nu(dt, dy) = K(X_{t-}, dy)dt$,

(c) *for all bounded measurable functions f, the following process is a martingale:*

$$M_t^f = f(X_t) - f(X_0) - \int_0^t Lf(X_s) ds.$$

22.2. **Exponential martingales for jump processes.** Let $(X_t)_{t\geq 0}$ be a pure jump Markov process taking values in a subset I of $\mathbb{R}^d$, with Lévy kernel K. Consider the Laplace transform

$$m(x,\theta) = \int_{\mathbb{R}^d} e^{\langle\theta,y\rangle} K(x,dy)$$

and assume that, for some $\eta_0 > 0$ and $C < \infty$,

$$\sup_{x\in I} \sup_{|\theta|\leq\eta_0} m(x,\theta) \leq C. \tag{22.1}$$

This implies in particular that K is bounded and that the following process is a martingale

$$M_t = \int_0^t \int_{\mathbb{R}^d} y(\mu - \nu)(ds,dy).$$

Note that

$$X_t = X_0 + M_t + \int_0^t b(X_s)ds,$$

where

$$b(x) = \int_{\mathbb{R}^d} yK(x,dy).$$

We obtain in this section an estimate on M in terms of K, which will be used to show, in a certain asymptotic regime, that M is small and so, when X_0 is close to x_0, X is close to the solution of the ODE

$$x_t = x_0 + \int_0^t b(x_s)ds.$$

Lemma 22.2.1. *Let K be a Borel measure on $\mathbb{R}^d$ and set*

$$m(\theta) = \int_{\mathbb{R}^d} e^{\langle\theta,y\rangle} K(dy).$$

Let $\eta_0 > 0$ and suppose that $m(\theta) \leq C$ whenever $|\theta| \leq \eta_0$. Then, for all $\eta \in (0,\eta_0)$ and $\delta = \eta_0 - \eta$,

$$|m''(\theta)| \leq 2Cd/\delta^2, \quad |\theta| \leq \eta.$$

Fix $\eta \in (0,\eta_0)$. Then there exists $A < \infty$ such that

$$|m''(x,\theta)| \leq A, \qquad x \in I, \quad |\theta| \leq \eta, \tag{22.2}$$

where $'$ denotes differentiation in θ. Define for $\theta \in (\mathbb{R}^d)^*$

$$\phi(x,\theta) = \int_{\mathbb{R}^d} \{e^{\langle\theta,y\rangle} - 1 - \langle\theta,y\rangle\} K(x,dy).$$

Then $\phi \geq 0$ and, for $|\theta| \leq \eta$, by the second-order mean value theorem,

$$\phi(x,\theta) = \int_0^1 m''(x,r\theta)(\theta,\theta)(1-r)dr$$

so

$$\phi(x,\theta) \leq \frac{1}{2}A|\theta|^2, \qquad x \in I, \quad |\theta| \leq \eta.$$

Proposition 22.2.2 (Exponential martingale inequality). *For all $\delta \in (0, A\eta t\sqrt{d}]$*

$$\mathbb{P}\left(\sup_{s\leqslant t}|M_s| > \delta\right) \leqslant (2d)e^{-\delta^2/(2Adt)}.$$

22.3. **Fluid limit for stopped processes.** Let $(X_t^N)_{t\geq 0}$ be a sequence of pure jump Markov processes, X^N taking its values in a measurable set $I^N \subseteq \mathbb{R}^d$ and having Lévy kernel K^N.

Let S be an open set in $\mathbb{R}^d$ and set $S^N = I^N \cap S$. We shall study, under certain hypotheses, the limiting behaviour of X^N as $N \to \infty$, up to the first time the process leaves S.

The set S may be thought of as arising as follows: there is some open set H such that $K^N(x,dy) = 0$ for $x \notin H$, so $(X_t^N)_{t\geq 0}$ stops on leaving H; we choose a further open set U according to our convenience, which should contain, with high probability, the whole path $(X_t^N)_{t\geq 0}$ when N is large, and serve to protect us from any singularities of the Lévy kernels K^N; then $S = H \cup U$.

Consider the Laplace transform

$$m^N(x,\theta) = \int_{\mathbb{R}^d} e^{\langle\theta,y\rangle}K^N(x,dy), \quad x \in S^N, \theta \in (\mathbb{R}^d)^*.$$

We assume that there is a constant $\eta_0 > 0$ such that

$$\sup_N \sup_{x\in S^N} \sup_{|\theta|\leqslant\eta_0} m^N(x,N\theta)/N < \infty. \tag{22.3}$$

Set

$$b^N(x) = \int_{\mathbb{R}^d} yK^N(x,dy).$$

We assume that there is a Lipschitz vector field b on S such that

$$\sup_{x\in S^N} |b^N(x) - b(x)| \to 0 \quad \text{as } N \to \infty. \tag{22.4}$$

We also assume

(22.5) $\qquad S$ has a Lipschitz boundary

so that b has an extension as a Lipschitz vector field $\tilde{b}$ on $\mathbb{R}^d$. Fix a point x_0 in the closure $\bar{S}$ of S and denote by $(x_t)_{t\geqslant 0}$ the unique solution to $\dot{x}_t = \tilde{b}(x_t)$ starting from x_0.

Assume that, for all $\delta > 0$,

$$\limsup_{N\to\infty} N^{-1}\log\mathbb{P}(|X_0^N - x_0| > \delta) < 0. \tag{22.6}$$

Fix $t_0 > 0$ and set

$$T^N = \inf\{t \geqslant 0 : X_t^N \notin S\} \wedge t_0$$

Theorem 22.3.1 (Fluid limit)**.** *Under assumptions (22.3), (22.4), (22.5), (22.6), we have, for all* $\delta > 0$,

$$\limsup_{N\to\infty} N^{-1} \log \mathbb{P}\big(\sup_{t \leq T^N} \big|X_t^N - x_t\big| > \delta\big) < 0. \tag{22.7}$$

Exercises

19.1 Denote by $\mathcal{P}_1$ the σ-algebra on $\Omega \times (0, \infty)$ generated by all continuous adapted processes and by $\mathcal{P}_2$ the σ-algebra on $\Omega \times (0, \infty)$ generated by all processes of the form

$$H_t = 1_{S<t\leq T}$$

where S and T are stopping times. Show that both $\mathcal{P}_1$ and $\mathcal{P}_2$ coincide with the previsible σ-algebra $\mathcal{P}$.

19.2 Let X be an adapted, real-valued process. Let $A \subseteq \mathbb{R}$. Set

$$T = \inf\{t \geq 0 : X_t \in A\}.$$

Show that T is a stopping time in each of the following cases:

(a) X is continuous and A is closed,
(b) X is right-continuous, $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous and A is open.

Find an example where X is continuous but T is not a stopping time.

19.3 Say that a process X is *progressively measurable* if, for all $t > 0$, the following map is measurable:

$$(\omega, s) \mapsto X(\omega, s) : (\Omega \times [0, t], \mathcal{F}_t \otimes \mathcal{B}[0, t]) \to (\mathbb{R}, \mathcal{B}).$$

Show that every right-continuous adapted process is progressively measurable.

Let T be a finite stopping time and let X be progressively measurable. Show that, for all $t > 0$, the map

$$\omega \mapsto (T(\omega), \omega) : (\{T \leq t\}, \mathcal{F}_t) \to (\Omega \times [0, t], \mathcal{F}_t \otimes \mathcal{B}[0, t])$$

is measurable and deduce that X_T is $\mathcal{F}_T$-measurable.

19.4 Show for $a \in L^1(\mathbb{R}^+)$ that the process A has total variation process V, where

$$A_t = \int_0^t a_s ds, \quad V_t = \int_0^t |a_s| ds.$$

19.5 Let A be a continuous finite variation process and let X be a continuous process. Show that

$$\lim_{n\to\infty} \sum_{k=0}^{[2^n t]-1} (A_{(k+1)2^{-n}} - A_{k2^{-n}})(X_{(k+1)2^{-n}} - X_{k2^{-n}}) = 0.$$

19.6 Let M be a cadlag L^2-bounded martingale starting from 0 and let $0 = T_0 \leq T_1 \leq \cdots \leq T_n \uparrow \infty$ be a sequence of stopping times. Show that

$$\mathbb{E}(M_\infty^2) = \sum_{n=0}^{\infty} \mathbb{E}[(M(T_{n+1}) - M(T_n))^2].$$

19.7 Show that any uniform limit of cadlag processes on $[0, \infty)$ is cadlag.

19.8

(a) Let Y be a random variable taking values ± 1, each with probability $1/2$. Set $\mathcal{F}_t = \{\emptyset, \Omega\}$ for $t \leq 1$, and $\mathcal{F}_t = \sigma(Y)$ for $t > 1$. Show that $\mathcal{M}$ contains only constants, so the isometry $M \mapsto M_\infty$ to $L^2(\mathcal{F}_\infty)$ is not onto.

(b) Show, on the other hand, that, if $(\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions then this isometry is onto.

19.9 Show that the set of simple processes $\mathcal{S}$ is a vector space.

19.10 Let M be a continuous local martingale starting from 0. Show that $\mathbb{E}(M_t^2) \leq \mathbb{E}([M]_t)$ for all $t \geq 0$ and that the following are equivalent:

(a) M is an L^2-bounded martingale,

(b) $\mathbb{E}([M]_\infty) < \infty$.

19.11 Let B and W be independent Brownian motions. Show that $[B]_t = t$ and $[B, W]_t = 0$ for all $t \geq 0$.

19.12 Let $U \subseteq \mathbb{R}^d$ be open and let $f : U \to \mathbb{R}$ be C^2 with $\Delta f = 0$ on U. Show that, if B is a Brownian motion in $\mathbb{R}^d$ and if $M_t = f(B_t)$, then $M = (M_t)_{t<\zeta}$ is a locally defined local martingale, where $\zeta = \inf\{t \geq 0 : B_t \notin U\}$.

19.13 Take $U = \mathbb{R}^3 \setminus \{(0,0,0)\}$ and $f(x) = |x|^{-1}$ in **19.12** and suppose that $B_0 = (1, 0, 0)$. Show

(a) $\zeta = \infty$ a.s.,

(b) M is bounded in L^2,

(c) $M_t \to 0$ as $t \to \infty$ a.s. and in L^1.

We have $\mathbb{E}(M_0) = 1 \neq 0 = \mathbb{E}(M_\infty)$. Why does this not contradict the L^2-martingale convergence theorem?

19.14 Let B be a Brownian motion in $\mathbb{R}^n$ and let $f \in C^{1,2}([0, \infty) \times \mathbb{R}^n)$. Set

$$M_t^f = f(t, B_t) - f(0, B_0) - \int_0^t \left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta \right) f(s, B_s) ds.$$

Show that M^f is a local martingale. Under what further conditions on f can we show that M^f is a martingale?

20.1 Show that, for suitably smooth coefficients, the following stochastic differential equations have the same solutions:

$$dX_t = \sigma(X_t) dB_t + b(X_t) dt,$$
$$\partial X_t = \sigma(X_t) \partial B_t + \tilde{b}(X_t) \partial t,$$

where $\tilde{b} = b - \frac{1}{2}(\partial \sigma / \partial x)\sigma$.

20.2 Suppose that X_t satisfies the stochastic differential equation

$$dX_t = \sigma(X_t)dB_t, \qquad X_0 = x_0,$$

where σ is Lipschitz and B_t is a Brownian motion. Show that, for some constant $C < \infty$, depending only on the Lipschitz constant of σ, X_t satisfies the estimate

$$\mathbb{E}\left[\sup_{s\le t}|X_s - x_0|^2\right] \le Cte^{Ct}|\sigma(x_0)|^2.$$

Discuss the tightness of this estimate with reference to the special cases $\sigma(x) = 1$ and $\sigma(x) = x$.

20.3 Let $a, b, c : \mathbb{R} \to \mathbb{R}$ be bounded with bounded derivatives and with $a > 0$. Suppose that $u \in C_b^{1,2}(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$ solves the Cauchy problem

$$\begin{cases} \partial u/\partial t = Lu & \text{in } \mathbb{R}^+ \times \mathbb{R} \\ u(0,\cdot) = f & \text{on } \mathbb{R} \end{cases}$$

where

$$Lu = \tfrac{1}{2}a(x)u'' + b(x)u' + c(x)u.$$

Set $\sigma = \sqrt{a}$ and define X by the SDE

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt, \qquad X_0 = x.$$

Define also $E_t = \exp\{\int_0^t c(X_s)ds\}$. Show that

$$u(t,x) = \mathbb{E}\big(E_t f(X_t)\big).$$

20.4 Suppose that σ, b and $\sigma_n, b_n, n = 1, 2, \dots$ are Lipschitz, with constant K independent of n. Suppose also that $\sigma_n \to \sigma$ and $b_n \to b$ uniformly. Define X and X^n by

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt, \qquad X_0 = x,$$
$$dX_t^n = \sigma_n(X_t^n)dB_t + b_n(X_t^n)dt, \qquad X_0^n = x.$$

Show that as $n \to \infty$

$$\mathbb{E}\left[\sup_{s\le t}|X_s^n - X_s|^2\right] \to 0.$$

21.1 Let N be a Poisson process of rate 1. Set $M_t = N_t - t$. Show that both M_t and $M_t^2 - t$ are martingales. Why does this not contradict Lévy's characterization of Brownian motion?

21.2 Consider the stochastic differential equation

$$dX_t = dB_t - \lambda X_t dt, \qquad X_0 = 0.$$

Find the joint distribution of $(X_t, \int_0^t X_s ds)$.

21.3 Consider the SDE in $\mathbb{R}$

$$dX_t = dB_t + b(X_t)dt, \qquad X_0 = x$$

where b is bounded measurable. Begin with a probability measure $\mathbb{P}$ and a Brownian motion X starting from x. Use Girsanov's theorem to find a new probability measure $\tilde{\mathbb{P}}$, absolutely continuous with respect to $\mathbb{P}$, such that if

$$B_t = X_t - \int_0^t b(X_s)ds$$

then $(B_t)_{0\le t\le 1}$ is a Brownian motion under $\tilde{\mathbb{P}}$. This is called solution by *transformation of drift*.

21.4 Let $f : \mathbb{C} \to \mathbb{C}$ be analytic and let $Z_t = X_t + iY_t$ where (X, Y) is a Brownian motion in $\mathbb{R}^2$. Use Itô's formula to show that $M = f(Z)$ is a local martingale (in $\mathbb{R}^2$). Show further that M is a time change of Brownian motion in $\mathbb{R}^2$.

Let $D = \{|z| \le 1\}$ and take $z \in D$. What is the hitting distribution for Z on ∂D in the case $Z_0 = 0$? By considering an analytic map $f : D \to D$ with $f(0) = z$, determine the hitting distribution for Z on ∂D when $Z_0 = z$.

21.5 Consider the stochastic differential equation

$$dX_t = \sigma(X_t)dB_t, \qquad X_0 = 0$$

where $\sigma : \mathbb{R} \to (0, \infty)$ is locally Lipschitz and $\sigma \equiv 1$ on $(-\infty, 0]$. Show by expressing the solution X as a time change of Brownian motion that X cannot explode.

21.6 Show how to construct a probability measure $\mathbb{P}$, a continuous process X and a $\mathbb{P}$-Brownian motion B adapted to the natural filtration of X such that

$$X_t = B_t + \int_0^t sgn(X_s)ds$$

for all $t \ge 0$, where

$$sgn(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Explain intuitively why $\mathbb{P}(X_t/t \to 1) = \frac{1}{2}$.

22.1 State the fluid limit theorem for a sequence of pure jump Markov processes in $\mathbb{R}^d$.

Consider a stochastic epidemic (S^N, I^N) in a population of N individuals, with infection rate λ and removal rate μ. Thus, $(S_t^N, I_t^N)_{t\ge 0}$ is a continuous-time Markov chain, taking values in $(\mathbb{Z}^+)^2$ where

$$\begin{aligned} (s,i) &\to (s-1, i+1) \quad \text{at rate} \quad \lambda s(i/N), \\ (s,i) &\to (s, i-1) \quad \text{at rate} \quad \mu i. \end{aligned}$$

Let $X_t^N = (S_t^N, I_t^N)/N$ be the proportion of susceptibles and infectives at time t and suppose that, for some $\alpha > 0$, X_0^N converges deterministically to $x_0 = (1-\alpha, \alpha)$ as

$N \to \infty$. Show that, for all $\delta > 0$,

$$\limsup_{N\to\infty} N^{-1} \log \mathbb{P}\left(\sup_{t\geq 0} |X_t^N - x_{t\wedge\tau}| > \delta\right) < 0,$$

where

$$\dot{x}_t^1 = -\lambda x_t^1 x_t^2, \quad \dot{x}_t^2 = (\lambda x_t^1 - \mu) x_t^2,$$

and where $\tau = \inf\{t \geq 0 : x_t^2 = 0\}$.

22.2 For $x = (x^1, x^2) \in \mathbb{R}^2 \setminus \{0\}$ set $n(x) = (-x^2, x^1)/|x|$ and consider the pure jump Markov process X in $\mathbb{R}^2$ having Lévy kernel

$$K(x, A) = \begin{cases} |x|, & \text{if } n(x) \in A, \\ 0, & \text{if } n(x) \notin A, \end{cases}$$

and starting from $X_0 = (a, 0)$, where $a > 0$. Show that

(a) for all a,

$$\mathbb{P}_a(|X_t| \to \infty \text{ as } t \to \infty) = 1,$$

(b) for all $\delta > 0$ and $t > 0$,

$$\limsup_{a\to\infty} a^{-1} \log \mathbb{P}_a\left(\sup_{s\leq t} |a^{-1} X_s - (\cos s, \sin s)| > \delta\right) < 0.$$

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