Stochastic Loewner Evolutions: Examples Sheet 2

1. Suppose that a function Φ is analytic near ξ, with $\Phi'(\xi) > 0$. Show that, as $z \to \xi$,

$$\frac{2\Phi'(\xi)^2}{\Phi(z) - \Phi(\xi)} - \Phi'(z)\frac{2}{z-\xi} \to -3\Phi''(\xi)$$

and

$$2\left(-\frac{\Phi'(\xi)^2\Phi'(z)}{(\Phi(z)-\Phi(\xi))^2} + \frac{\Phi'(z)}{(z-\xi)^2} - \frac{\Phi''(z)}{z-\xi}\right) \to \frac{1}{2}\frac{\Phi''(\xi)^2}{\Phi'(\xi)} - \frac{4}{3}\Phi'''(\xi).$$

It is worth reducing to the case $\xi = \Phi(\xi) = 0$ and $\Phi'(\xi) = 1$ (how?) before starting to compute.

2. Let γ be an $SLE(\kappa)$. Define for $r > 0$ and $s \geqslant 0$

$$(\sigma_r\gamma)_t = r^{-1}\gamma_{r^2t}, \quad (\theta_s\gamma)_t = g_s(\gamma_{s+t}) - \xi_s.$$

Express the Loewner tranforms of $\sigma_r\gamma$ and $\theta_s\gamma$ in terms of the Loewner transform of γ and hence show that both $\sigma_r\gamma$ and $\theta_s\gamma$ are also $SLE(\kappa)$.

3. Let $(X_t)_{t<\zeta}$ be the maximal solution to the Bessel stochastic differential equation

$$dX_t = dB_t + \frac{a}{X_t}dt, \quad X_0 = x,$$

where B is a Brownian motion. Fix $r > 0$ and set $Y_t = r^{-1}X_{r^2t}$. Show that

$$dY_t = dW_t + \frac{a}{Y_t}dt,$$

for some Brownian motion W.

Suppose that $a \in (0, 1/2)$ so that $\zeta < \infty$, almost surely. Define for $r \in (0, x]$,

$$T(r) = \inf\{t \geqslant 0 : X_t = r\}.$$

Show that the random variables

$$A_n = \int_{T(2^{-n+1}x)}^{T(2^{-n}x)} \frac{1}{X_t^2}dt, \quad n \geqslant 1,$$

are independent and identically distributed.

Suppose now that $a \in (1/2, \infty)$. We know that $\zeta = \infty$ almost surely. Show that $\inf_{t\geqslant 0} X_t > 0$ almost surely.

Finally, consider the case $a = 1/2$. Show that $\zeta = \infty$ but $\inf_{t\geqslant 0} X_t = 0$ almost surely.

4.

5.