Stochastic Loewner Evolutions: Examples Sheet 1

1. Suppose that f is an analytic function, defined on a neighbourhood of 0, with $f(0) = f'(0) = 0$ and $f''(0) = 1$. By considering the values of $\arg f$ round a small circle centred at 0, show that f is not a bijection. Deduce that, for any open set D in $\mathbb{C}$ and any analytic function Φ defined and one-to-one on D, we must have: (i) $\Phi(D)$ is an open set, (ii) Φ is a conformal map, (iii) the inverse map $\Phi^{-1} : \Phi(D) \to D$ is analytic.

2. Suppose that σ is a conformal isomorphism $\mathbb{H} \to \mathbb{H}$ fixing 0 and ∞. Show that there exists $\lambda \in (0, \infty$ such that $\sigma(z) = \lambda z$ for all z.

3. Let D be a simply connected domain in $\mathbb{C}$ and let $\Phi : D \to \mathbb{D}$ be a conformal isomorphism. Define a metric d on D by $d(z, w) = |\Phi(z) - \Phi(w)|$. Say that a sequence $(z_n)_{n \geqslant 1}$ in D is Φ-Cauchy if it is Cauchy for the metric d. Show that this property is in fact independent of the choice of Φ.

4. Use Itô's formula for C^2 functions to show that, for a proper complex domain D, if B is a complex Brownian motion starting from $z \in D$, and if u is a continuous function on $\bar{D}$ which is harmonic in D, then

$$u(z) = \mathbb{E}[u(B_{T_D})],$$

where $T_D = \inf\{t \geqslant 0 : B_t \notin D\}$.

5. Let D be a simply connected proper complex domain, whose boundary is a Jordan curve (a simple closed curve) ∂D. Let $z \in D$ and let $b_1, b_2, b_3 \in \partial D$ be distinct and positively ordered. Show that the conformal isomorphism provided by the Riemann mapping theorem $\Phi : D \to \mathbb{D}$ can be chosen to satisfy any one of the following three conditions, each of which then determines Φ uniquely:

(i) $\Phi(z) = 0$ and $\Phi'(z) > 0$;

(ii) $\Phi(z) = 0$ and $\Phi(b_1) = 1$;

(iii) $\Phi(b_1) = 1, \Phi(b_2) = i$ and $\Phi(b_3) = -1$.

6. Use the conformal invariance of harmonic measure to deduce from the case $z = 0$ the following formula for the hitting density of Brownian motion on the unit circle, starting from general $z \in \mathbb{D}$,

$$h_\Delta(z, t) = \frac{1}{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2}, \quad 0 \leqslant t < 2\pi.$$

Find an analogous formula for the case $|z| > 1$.

7. Show that there is only one conformal automorphism f of the upper half-plane $\mathbb{H}$ such that $f(z) - z \to 0$ as $z \to \infty$. Prove the uniqueness assertion of Proposition 4.5: *for a compact $\mathbb{H}$-hull K, there is at most one conformal isomorphism $g_K : \mathbb{H} \setminus K \to \mathbb{H}$ such that $g_K(z) - z \to 0$ as $z \to \infty$.*

Verify also that, if Φ is defined in a neighbourhood of 0 in $\mathbb{C}$, with

$$\Phi(z) = z + bz^2 + cz^3 + O(|z|^4)$$

as $z \to 0$, and if $g(z) = -1/\Phi(-1/z) - b$, then, as $z \to \infty$,

$$g(z) = z + \frac{b^2 - c}{z} + O(|z|^{-2}).$$

8. Show that $\operatorname{hcap}(K) \leqslant \operatorname{rad}(K)^2$ for all compact $\mathbb{H}$-hulls K. For which K does equality hold? Show also that there is a constant $C < \infty$ such that, for all compact $\mathbb{H}$-hulls K, we have

$$|z - g_K(z)| \leqslant C \operatorname{rad}(K), \quad z \in \mathbb{H} \setminus K$$

and, for all $\xi \in \bar{K} \cap \mathbb{R}$ and all $z \in \mathbb{H}$ with $|z - \xi| \geqslant 4 \operatorname{rad}(K)$,

$$\left| g_K(z) - z - \frac{\operatorname{hcap}(K)}{z - \xi} \right| \leqslant \frac{2C \operatorname{rad}(K) \operatorname{hcap}(K)}{|z - \xi|^2}.$$

9. Let K be a compact $\mathbb{H}$-hull. Show that

$$\operatorname{hcap}(K) = \lim_{y \to \infty} y \mathbb{E}_{iy}(\operatorname{Im}(B_T)),$$

where B is a complex Brownian motion starting from iy and $T = \inf\{t \geqslant 0 : B_t \notin \mathbb{H} \setminus K\}$.

10. Let $\gamma : [0, \infty) \to \bar{\mathbb{H}}$ be a simple path with $\gamma(0) = 0$, $\gamma_t \in \mathbb{H}$ for all $t > 0$ and $\operatorname{Im}(\gamma_t) \to \infty$ as $t \to \infty$. Set $K_t = \{\gamma_s : 0 < s \leqslant t\}$. Show that $(K_t)_{t \geqslant 0}$ is a strictly increasing family of compact $\mathbb{H}$-hulls, with $\operatorname{hcap}(K_t) \to \infty$ as $t \to \infty$, and having the local growth property.

11. Show that, for the solution flow $(g_t)_{t \geqslant 0}$ of the Loewner differential equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \xi_t}, \quad g_0(z) = z,$$

we have, as $z \to \infty$,

$$z(g_t(z) - z) \to 2t.$$

12. Let $(K_t)_{t \geqslant 0}$ be a family of compact $\mathbb{H}$-hulls having the local growth property. Show that the associated Loewner transform $(\xi_t)_{t \geqslant 0}$ is continuous.

13. Consider the family of compact $\mathbb{H}$-hulls $(K_t)_{t \geqslant 0}$ generated by the path

$$\gamma_t = \begin{cases} \frac{it}{1-t+it}, & t \leqslant 1, \\ 1 + i(t-1), & t > 1. \end{cases}$$

Does $(K_t)_{t \geqslant 0}$ have the local growth property? Justify your answer.

14. Let γ be an $SLE(\kappa)$. Define for $r > 0$ and $s \geqslant 0$

$$(\sigma_r \gamma)_t = r^{-1} \gamma_{r^2 t}, \quad (\theta_s \gamma)_t = g_s(\gamma_{s+t}) - \xi_s.$$

Express the Loewner tranforms of $\sigma_r \gamma$ and $\theta_s \gamma$ in terms of the Loewner transform of γ and hence show that both $\sigma_r \gamma$ and $\theta_s \gamma$ are also $SLE(\kappa)$.