5. SLE(κ)

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which is defined a Brownian motion $(B_t)_{t\geq 0}$.

Fix $\kappa \in [0,\infty)$ and set $\xi_t = \sqrt{\kappa}\, B_t$.

Thus $(\xi_t)_{t\geq 0}$ is Brownian motion of diffusivity κ.

Construct from $(\xi_t)_{t\geq 0}$, as in §4,

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \xi_t}, \quad g_0(z) = 0$$

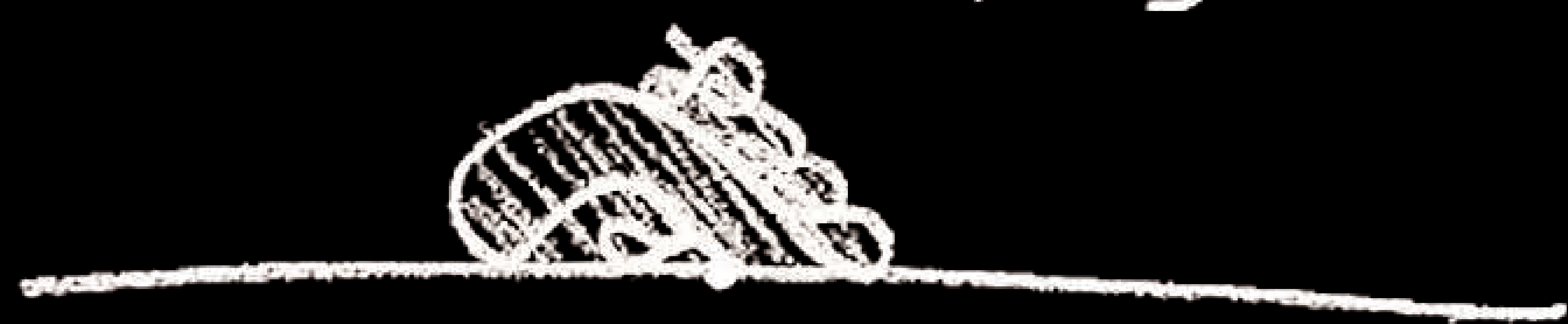
$g_t : H_t \to \mathbb{H}$ conformal isomorphism

$K_t = \mathbb{H} \setminus H_t$.

Say that a continuous function $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ <u>generates</u> $(K_t)_{t\geq 0}$ if

H_t <u>is the connected component of</u> $\mathbb{H}\setminus\gamma[0,t]$ <u>containing</u> ∞.

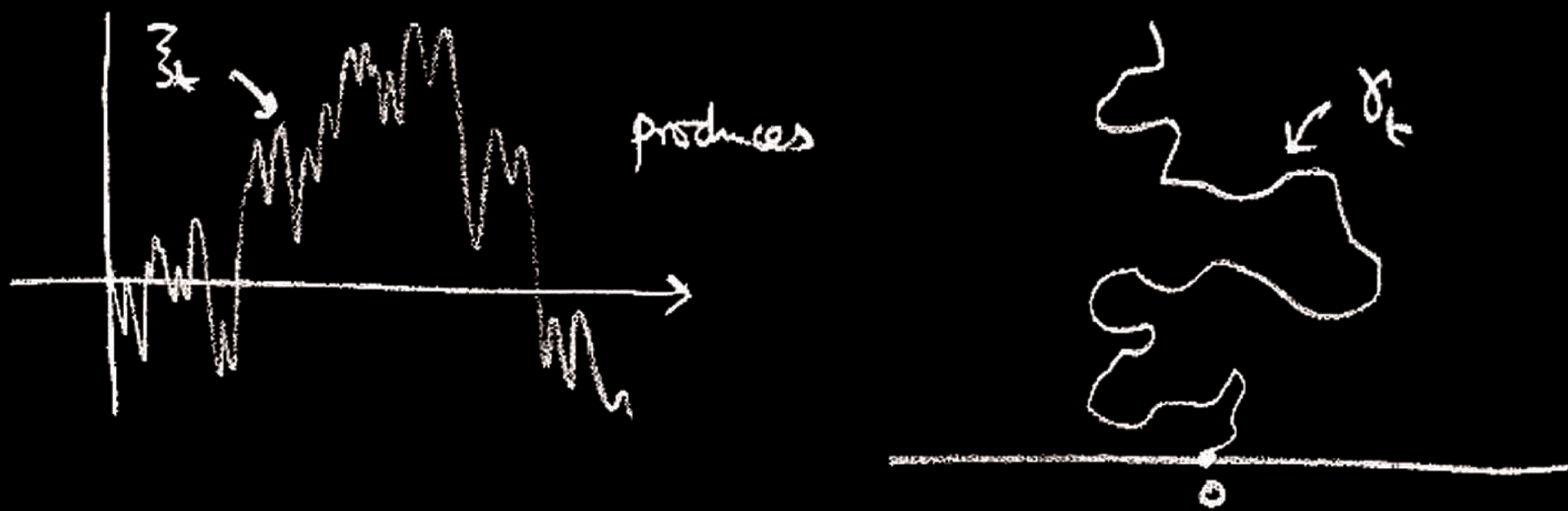
Any $(K_t)_{t\geq 0}$ so obtained is a family of compact $\mathbb{H}$-hulls having the local growth property.



<u>Theorem 5.1</u>

For all $\kappa \in [0,\infty)$, taking $\xi_t = \sqrt{\kappa}\, B_t$, there is a unique continuous random process $(\gamma_t)_{t\geq 0}$ in $\overline{\mathbb{H}}$ generating $(K_t)_{t\geq 0}$

The process $(\gamma_t)_{t\geq 0}$ is called a <u>stochastic Loewner evolution of parameter</u> κ, or <u>SLE</u>(κ) for short.



SLE inherits from Brownian motion a scaling property and a Markov property. Define for $r > 0$ and $s \geqslant 0$

$$(\sigma_r \xi)_t = r^{-1} \xi_{r^2 t}, \qquad (\sigma_r \gamma)_t = r^{-1} \gamma_{r^2 t}$$

and

$$(\theta_s \xi)_t = \xi_{t+s} - \xi_s, \qquad (\theta_s \gamma)_t = g_s(\gamma_{s+t}) - \xi_s.$$

Then $\sigma_r \gamma$ and $\theta_s \gamma$ have Loewner transforms $\sigma_r \xi$ and $\theta_s \xi$ respectively.

We know that the distributions of $\sigma_r \xi$ and $\theta_s \xi$ do not depend on r and s, with $\theta_s \xi$ independent of $(\xi_u : u \leq s)$. Hence analogous statements hold for $\sigma_r \gamma$ and $\theta_s \xi$

6. Phases of SLE

The behaviour of SLE(κ) changes dramatically as the parameter κ is varied. This may initially seem surprizing, as scaling a Brownian motion is a simple transformation producing no new effects.

Lemma 6.1

Fix $a \in (0,\infty)$ and consider, for each $x \in (0,\infty)$, the unique maximal solution $(X_t(x))_{t<\zeta(x)}$ in $(0,\infty)$ to

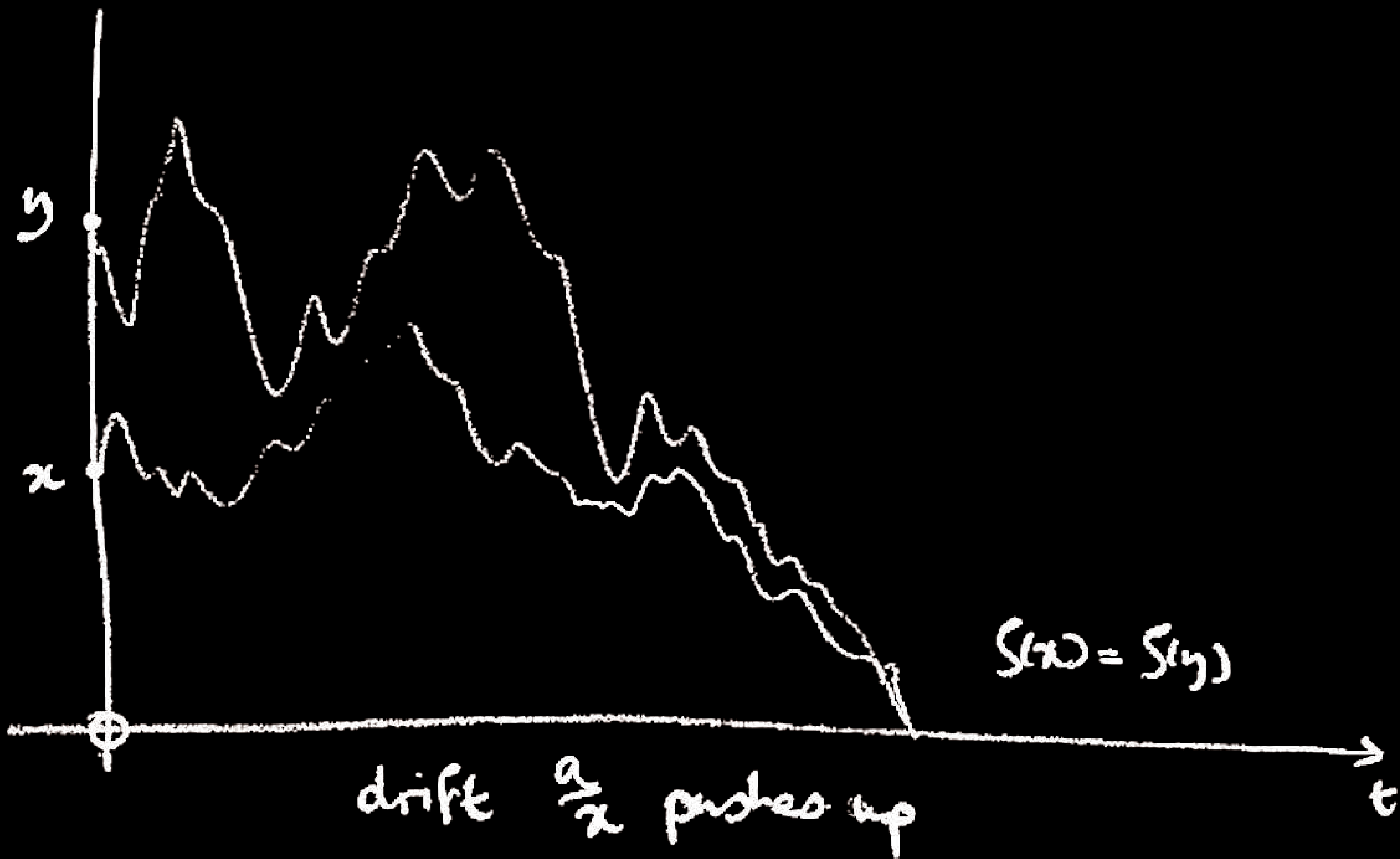
$$dX_t = dB_t + \frac{a}{X_t}\,dt, \quad X_0 = x$$

Then, for all $x, y \in (0,\infty)$ with $x<y$,

- for $a \in (0, \frac{1}{4}]$, $\mathbb{P}(\zeta(x) < \zeta(y) < \infty) = 1$;
- for $a \in (\frac{1}{4}, \frac{1}{2})$, $\mathbb{P}(\zeta(x) < \infty) = 1$, $\mathbb{P}(\zeta(x) < \zeta(y)) = \phi\left(\frac{y-x}{y}\right)$;
- for $a \in [\frac{1}{2}, \infty)$, $\mathbb{P}(\zeta(x) < \infty) = 0$.

Here, for $a \in (\frac{1}{4}, \frac{1}{2})$, $\theta \in [0,1]$, ϕ is given by

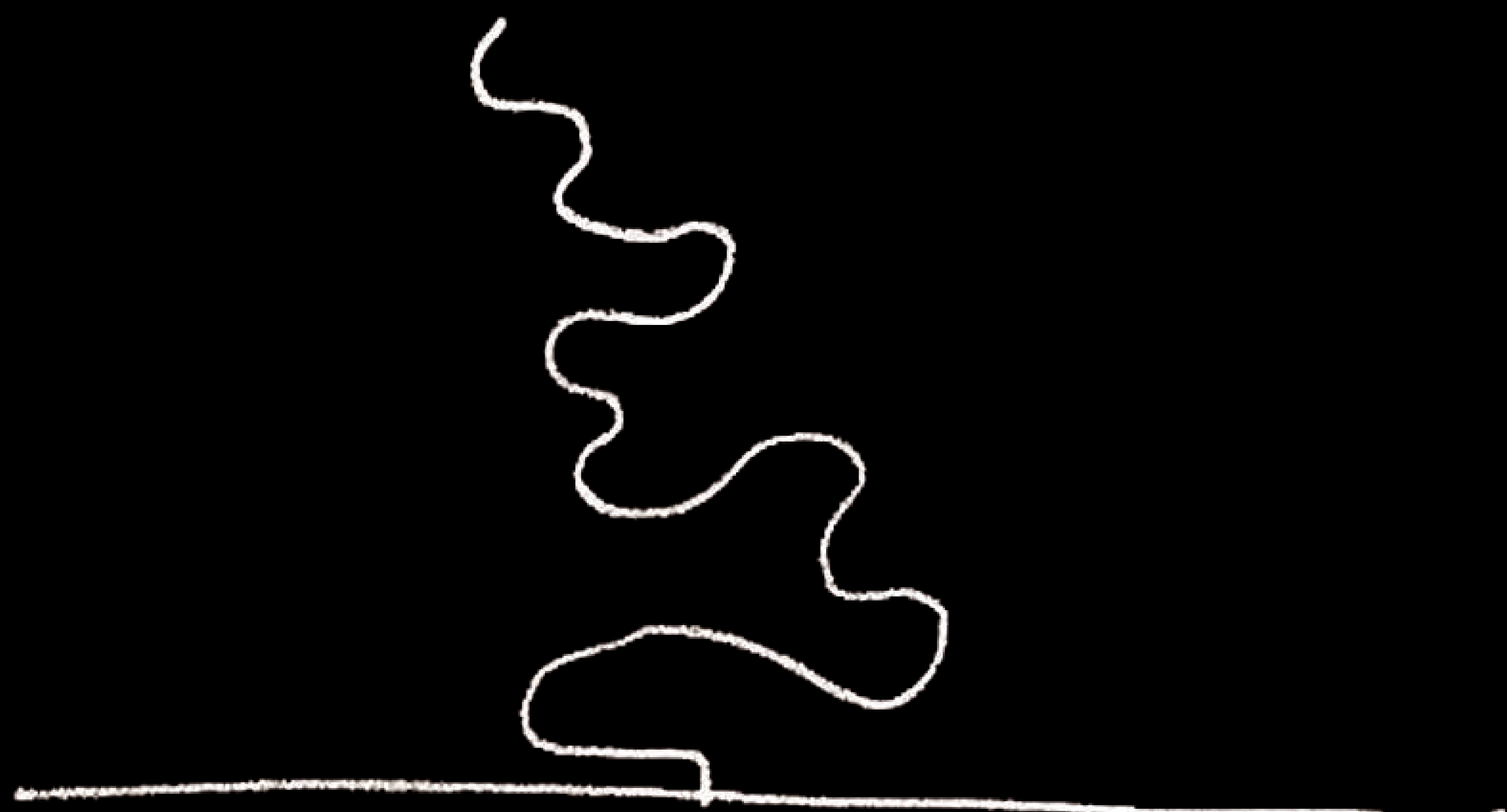
$$\phi(\theta) \propto \int_0^\theta \frac{du}{u^{2-4a}(1-u)^{2a}}, \quad \phi(1) = 1.$$


y
x
S(x) = S(y)
t
drift a/x pushes up

Proposition 6.2

Let γ be an $SLE(\kappa)$, with $\kappa \in [0,4]$.

Then $\gamma(0,\infty) \subseteq \mathbb{H}$ and γ is a simple curve, almost surely.



Proposition 6.3

Let γ be an SLE(κ), with $\kappa \in [0,4]$.

Then $\gamma_t \to \infty$ as $t \to \infty$, almost surely.

Proposition 6.4

Let γ be an SLE(κ), with $\kappa \in (4,8)$.

Then γ is neither simple, nor space-filling,

but $\mathrm{dist}(0, H_t) \to \infty$ as $t \to \infty$.

this behaviour is called swallowing

Proposition 6.5

Let γ be an $SLE(\kappa)$, with $\kappa \in [8, \infty)$.

Then, almost surely, $\gamma[0,\infty) = \mathbb{H}$ and $\gamma_t \to \infty$ as $t \to \infty$.