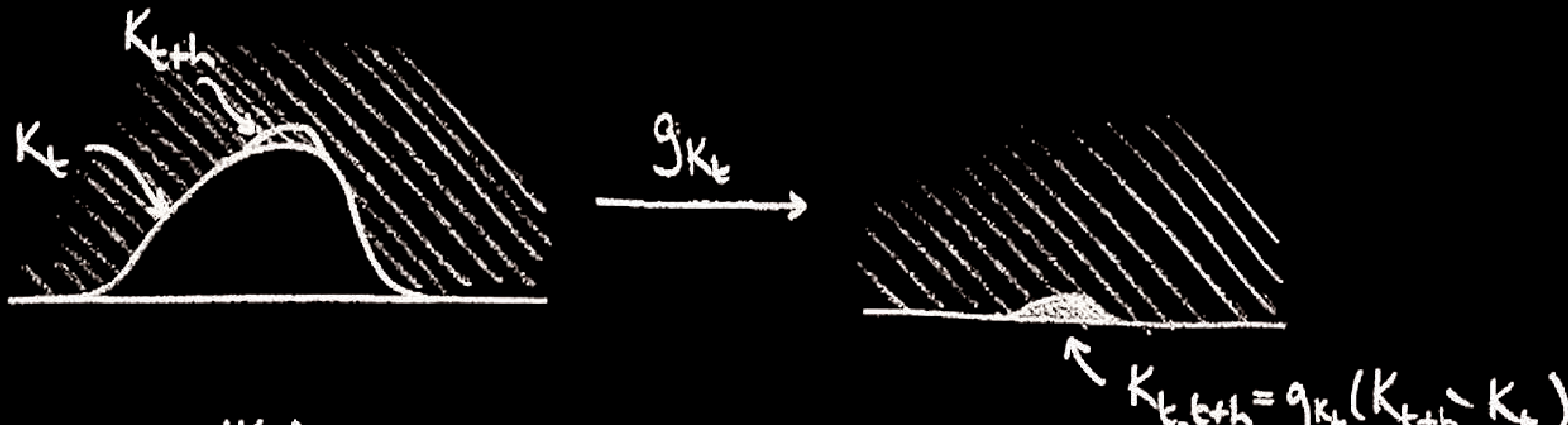
4. Loewner transforms

Let $(K_t)_{t\geq 0}$ be a strictly increasing family of compact $\mathbb{H}$-hulls, with $\text{hcap}(K_t) \to \infty$ as $t \to \infty$.



Say that $(K_t)_{t\geq 0}$ has the <u>local growth property</u> if

$$\text{rad}(K_{t,t+h}) \to 0 \text{ as } h \downarrow 0, \text{ uniformly on compacts in } t.$$

For such a family, by compactness, for each $t \geq 0$,

$$\bigcap_{h>0} \bar{K}_{t,t+h} = \{\xi_t\} \quad \text{for some } \xi_t \in \mathbb{R}.$$

Using the estimate

$$|z - g_K(z)| \leq C \operatorname{rad}(K)$$

it can be shown that $(\xi_t)_{t\geq 0}$ is continuous.

The process $(\xi_t)_{t\geq 0}$ is called the <u>Loewner transform</u> of $(K_t)_{t\geq 0}$.

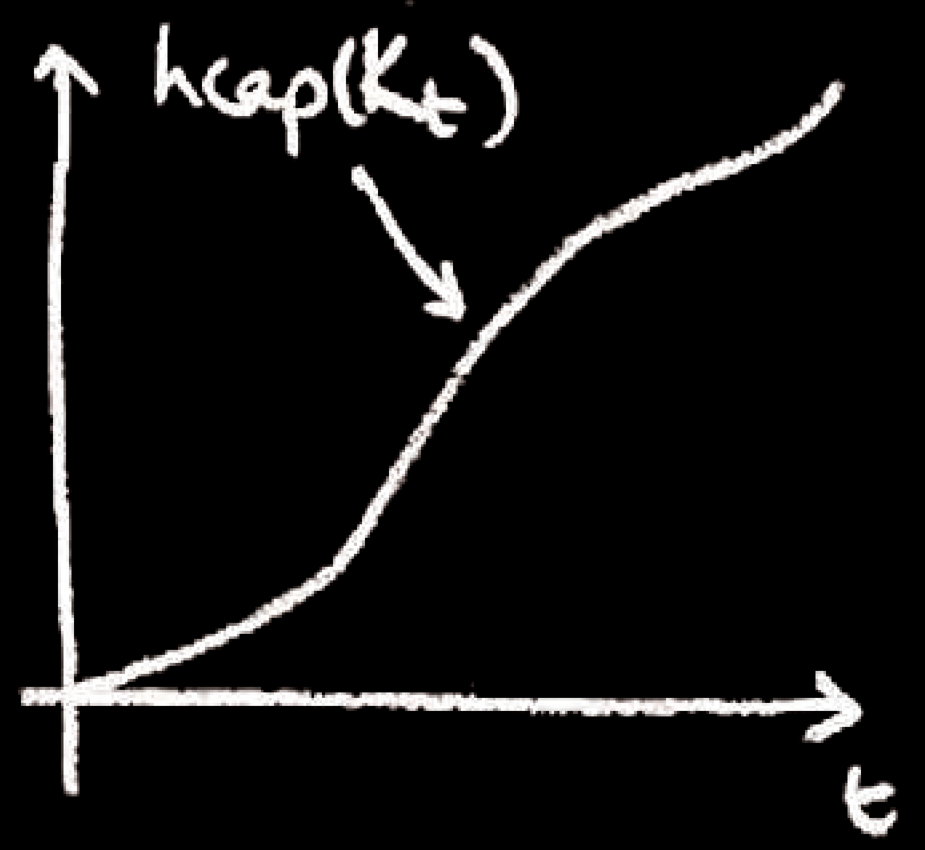
We have assumed

$$K_s \subsetneqq K_t \quad \text{for } s<t \quad \text{and} \quad \mathrm{hcap}(K_t) \to \infty \quad \text{as } t \to \infty.$$

Also

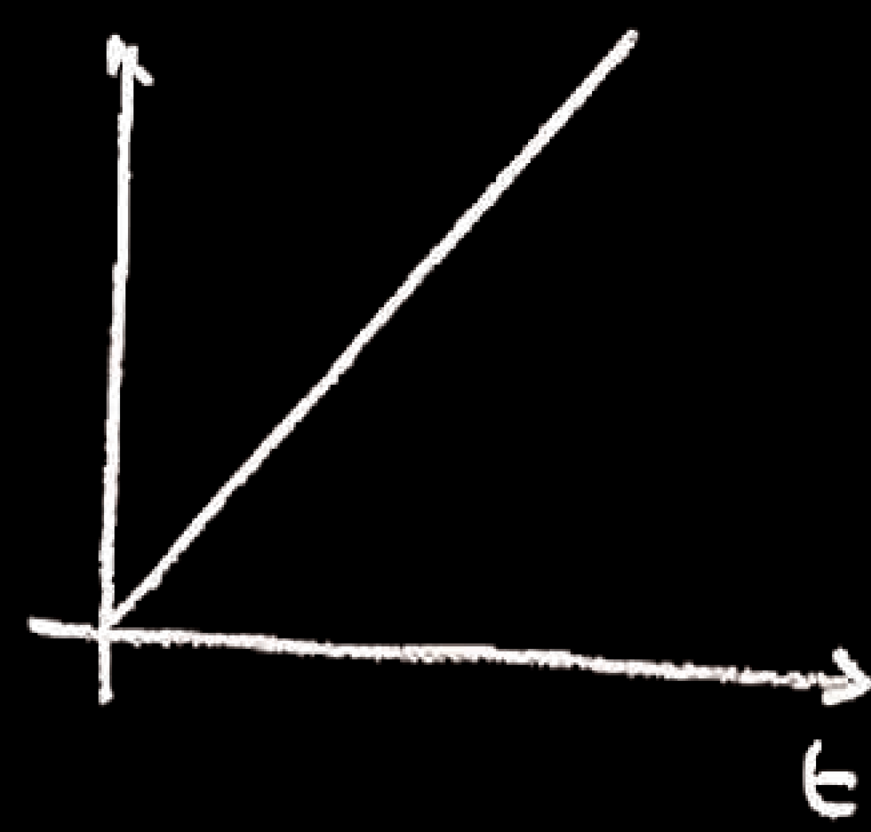
$$\mathrm{hcap}(K_{t+h}) - \mathrm{hcap}(K_t) = \mathrm{hcap}(K_{t,t+h}) \leq \mathrm{rad}(K_{t,t+h})^2 \to 0.$$

So $t \mapsto \mathrm{hcap}(K_t)$ is a homeomorphism of $[0,\infty)$,



and by a reparametrization we may assume that

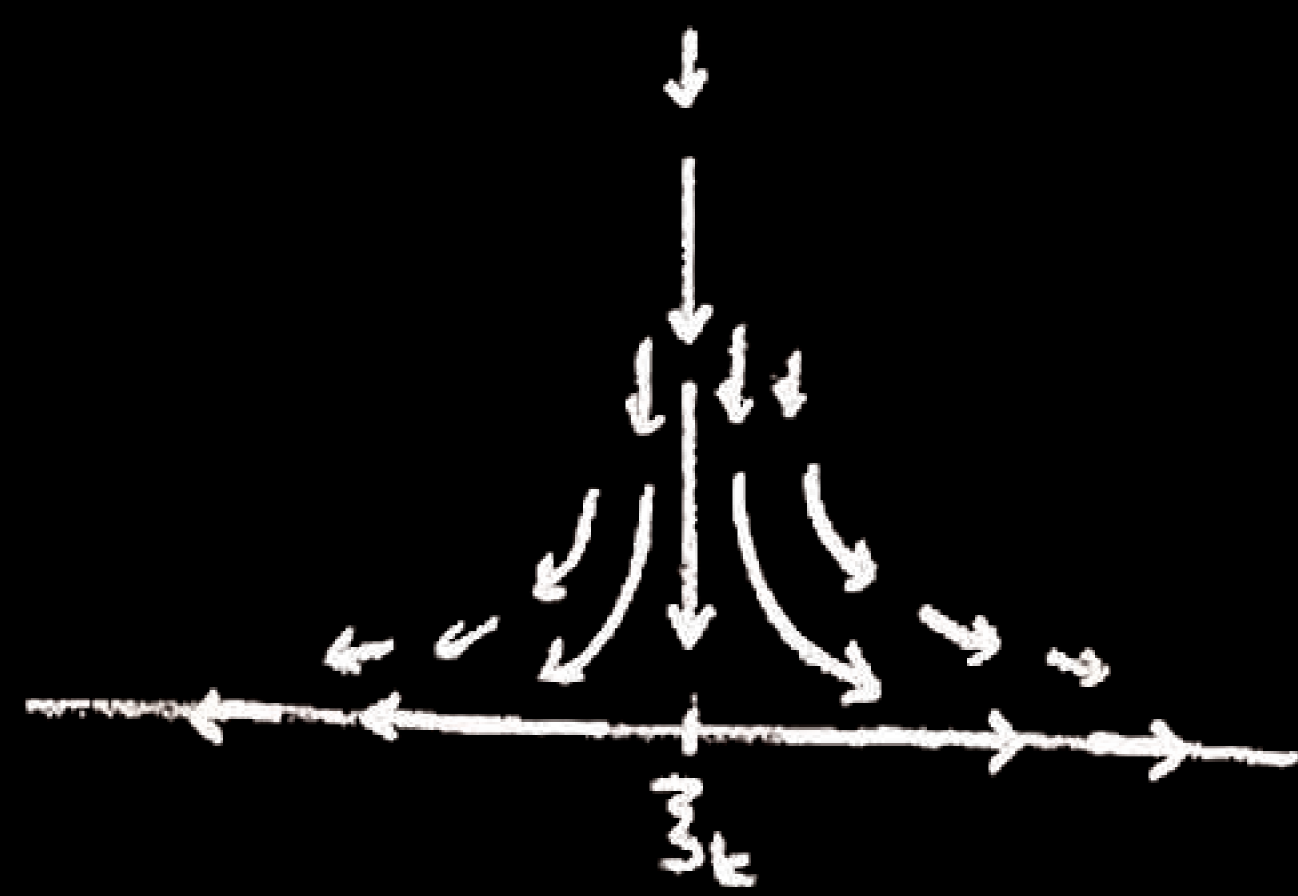
$$\mathrm{hcap}(K_t) = 2t.$$



Fix a continuous real-valued function $(\xi_t)_{t \geq 0}$.

Define a time-dependent vector field b on $\mathbb{H}$

$$b(t,z) = \frac{2}{z-\xi_t} = \frac{2(x-\xi_t - iy)}{|z-\xi_t|^2}$$



Note $b(t,\cdot)$ is Lipschitz on $\{z \in \mathbb{H} : |z-\xi_t| \geq \varepsilon\}$, uniformly in $t \geq 0$, for all $\varepsilon > 0$.

Also $\operatorname{Im} b(t,z) < 0$ for all $t \geq 0$, $z \in \mathbb{H}$.

The differential equation

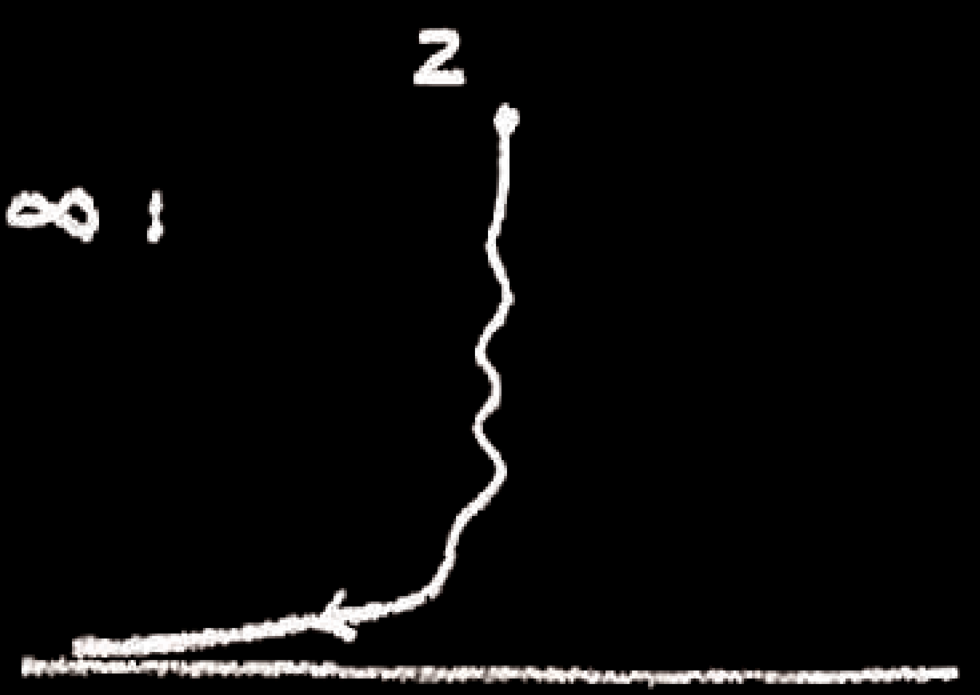
$$\dot{z}_t = b(t, z_t) = \frac{2}{z_t - \xi_t}, \qquad z_0 = z$$

has a unique maximal solution $(z_t)_{t<\zeta}$ in $\mathbb{H}$

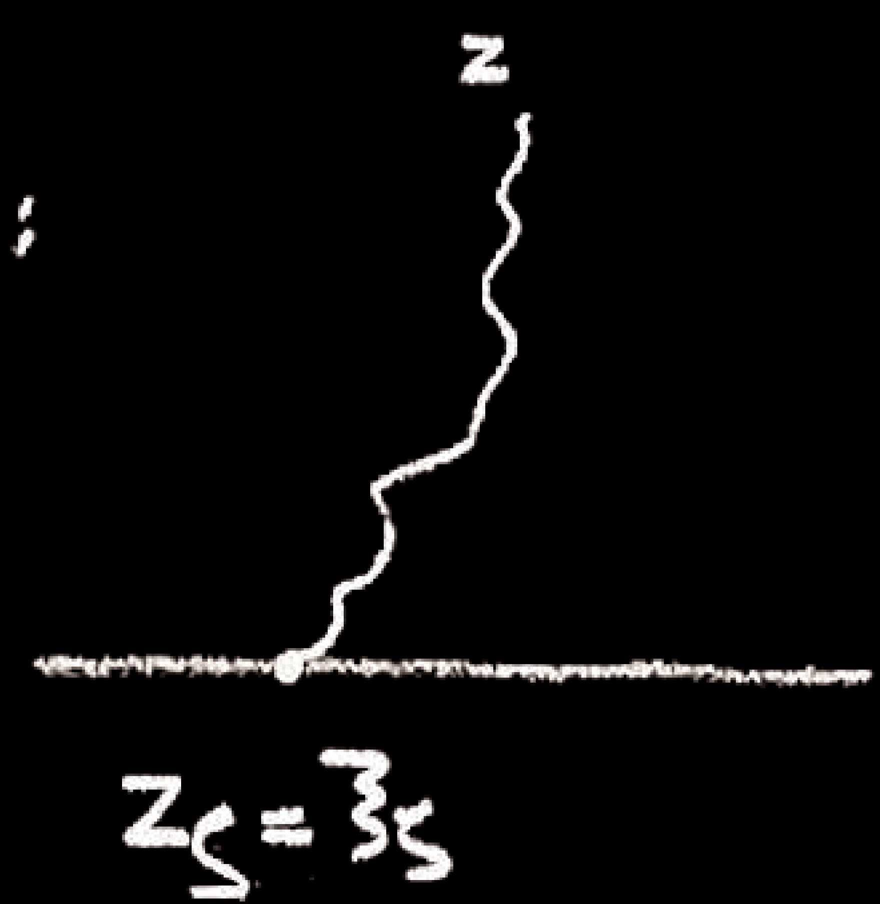
Moreover, if $\zeta < \infty$, then

$$\mathrm{Re}(z_t) - \xi_t \longrightarrow 0, \quad \mathrm{Im}(z_t) \downarrow 0 \qquad \text{at } t \uparrow \zeta.$$

$\zeta = \infty$:



$\zeta < \infty$:



We write $z_t = g_t(z)$ and $\zeta = \zeta(z)$.

Set $H_t = \{ z \in \mathbb{H} : \zeta(z) > t \}$. Then $g_t : H_t \to \mathbb{H}$.

Call $(g_t)_{t \geq 0}$ the <u>solution flow</u>.

<u>Lemma 4.1</u>

Set $R_t = \max \{ \sqrt{t}, \sup_{s \leq t} |\xi_s| \}$.

For $|z| \geq 4R_t$ and $s \leq t$, we have $|g_s(z) - z| \leq R_t$.

In particular $\zeta(z) > t$.

Proposition 4.2

Let $(K_t)_{t\geq 0}$ be an increasing family of compact $\mathbb{H}$-hulls, parametrized so that $\operatorname{hcap}(K_t) = 2t$ for all t. Define for $z \in \mathbb{H}$

$$\zeta(z) = \inf\{t \geq 0 : z \in K_t\}, \quad Z_t = g_t(z) = g_{K_t}(z), \quad t < \zeta(z).$$

Then $(Z_t)_{t<\zeta(z)}$ is differentiable, with $\dot{Z}_t = \dfrac{2}{Z - \xi_t}$, where $(\xi_t)_{t\geq 0}$ is the Loewner transform of $(K_t)_{t\geq 0}$.

Gronwall's lemma

Let $f : [0,T] \to \mathbb{R}$ be integrable.

Suppose

$$f(t) \leq A + B\int_0^t f(s)\,ds, \qquad 0 \leq t \leq T.$$

Then

$$f(T) \leq Ae^{BT}.$$

Continuity of solution flows in initial data

$$\dot{g}_t(z) = b(g_t(z))$$

$$|g_t(z) - g_t(z')| \leqslant |z - z'| + K \int_0^t |g_s(z) - g_s(z')| \, ds$$

so, by Gronwall's lemma,

$$|g_t(z) - g_t(z')| \leqslant |z - z'| \, e^{Kt}$$

(We used $|b(g_t(z)) - b(g_t(z'))| \leqslant K |g_t(z) - g_t(z')|$.

We shall also use

$$|b(g_t(z+h)) - b(g_t(z)) - b'(g_t(z))(g_t(z+h) - g_t(z))|$$
$$\leqslant K_2 |g_t(z+h) - g_t(z)|^2$$

Differentiability of solution flows in initial data

Fix z and define $(z'_t)_{t\geq 0}$ by $\dot{z}'_t = b'(z_t)z'_t$, $z'_0 = 1$.

(Thus $z'_t = \exp \int_0^t b'(z_s)\,ds$.)

Then

$$|g_t(z+h) - g_t(z) - hz'_t| = \left| \int_0^t \left(b(g_s(z+h)) - b(g_s(z)) - hb'(z_s)z'_s \right) ds \right|$$

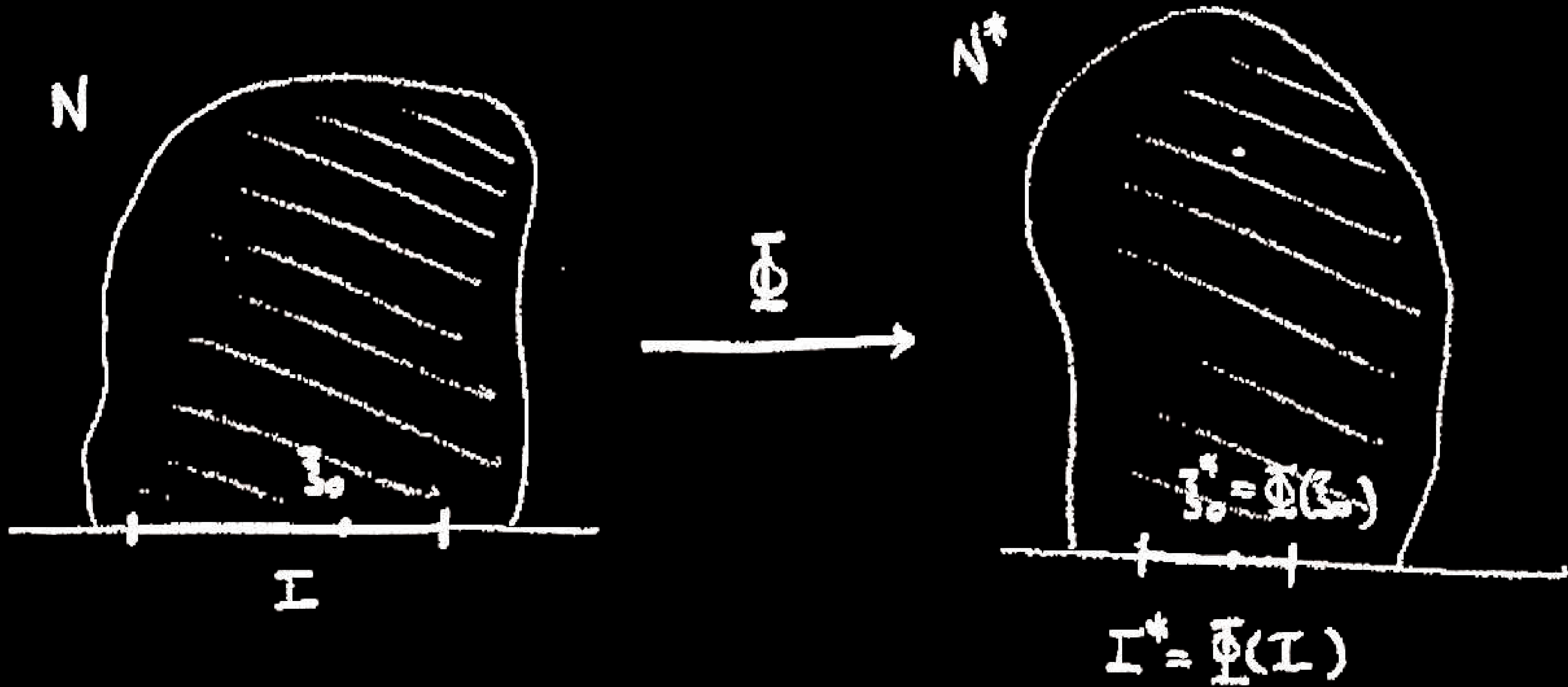
$$\leq \int_0^t | b(g_s(z+h)) - b(g_s(z)) - b'(z_s)(g_s(z+h) - g_s(z)) |\,ds$$

$$+ \int_0^t |b'(z_s)|\,|g_s(z+h) - g_s(z) - hz'_s|\,ds$$

$$\leq |h|^2 t e^{2Kt} K_2 + K \int_0^t |g_s(z+h) - g_s(z) - hz'_s|\,ds$$

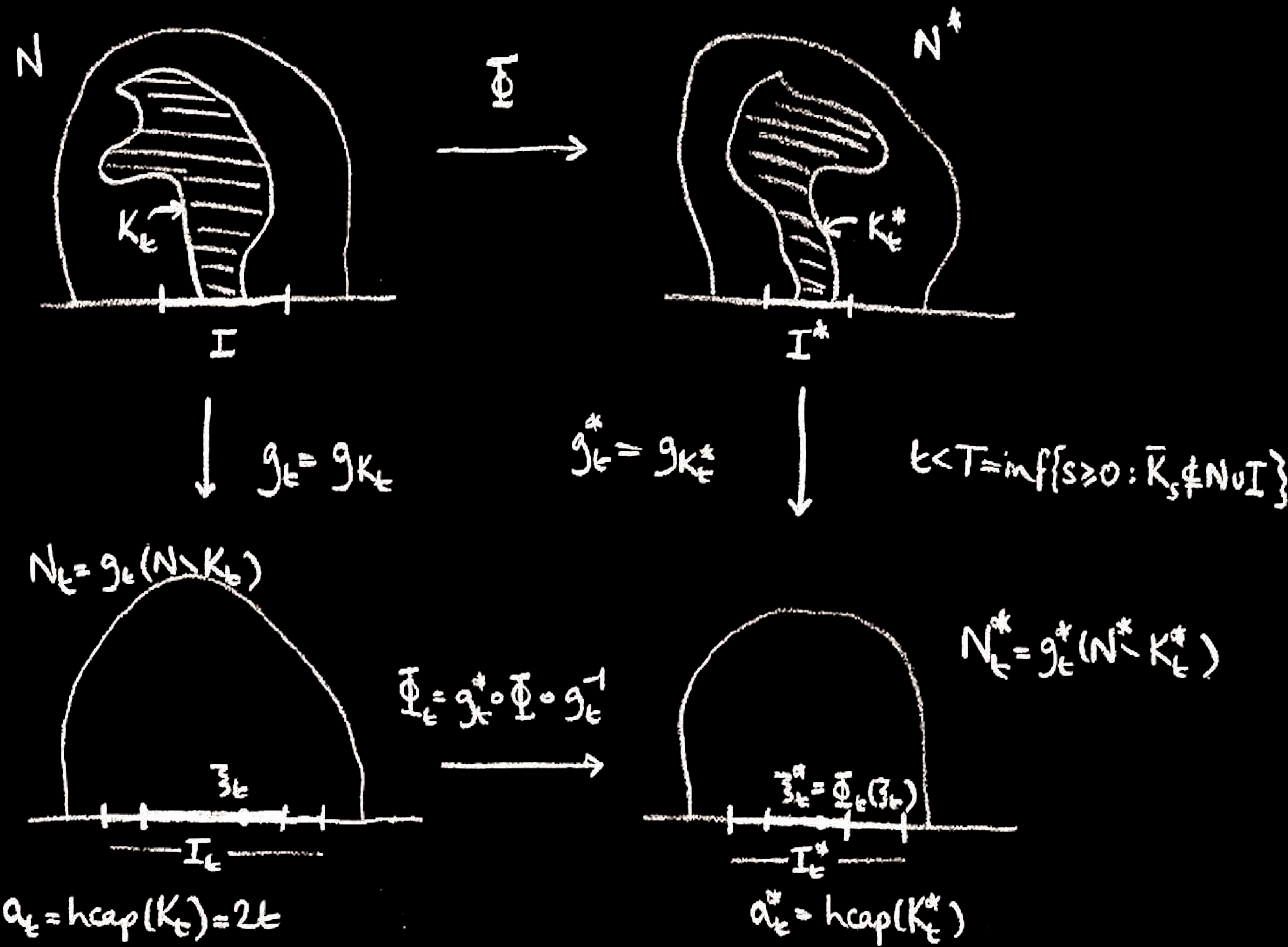
so, by Gronwall's lemma, ...

Evolution of conformal isomorphisms



Note that Φ extends analytically to I by reflection.

(We assume that N is a neighbourhood of I in $\mathbb{H}$)


N
Φ
N^*
K_t
K_t^*
I
I^*
$g_t = g_{K_t}$
$g_t^* = g_{K_t^*}$
$t < T = \inf\{s \geq 0 : \bar{K}_s \not\subset N \cup I\}$
$N_t = g_t(N \setminus K_t)$
$\Phi_t = g_t^* \circ \Phi \circ g_t^{-1}$
$N_t^* = g_t^*(N^* \setminus K_t^*)$
ξ_t
$\xi_t^* = \Phi_t(\xi_t)$
I_t
I_t^*
$a_t = \mathrm{hcap}(K_t) = 2t$
$a_t^* = \mathrm{hcap}(K_t^*)$

Proposition 4.3

The family of compact $\mathbb{H}$-hulls $(K_t^*)_{t<T}$ has the local growth property, with Loewner transform $(\xi_t^*)_{t<T}$.

Moreover the map $t \longmapsto a_t^*$ is differentiable on $[0,T)$, with

$$\dot{a}_t^* = 2\,\Phi_t'(\xi_t)^2 .$$

Proposition 4.4

The map $t \longmapsto \Phi_t$ on $[0,T)$ is a C^1 map of analytic functions, with

$$\dot{\Phi}_t(\zeta_t) = -3\Phi_t''(\zeta_t), \qquad \dot{\Phi}_t'(\zeta_t) = \frac{1}{2}\frac{\Phi_t''(\zeta_t)^2}{\Phi_t'(\zeta_t)} - \frac{4}{3}\Phi_t'''(\zeta_t).$$