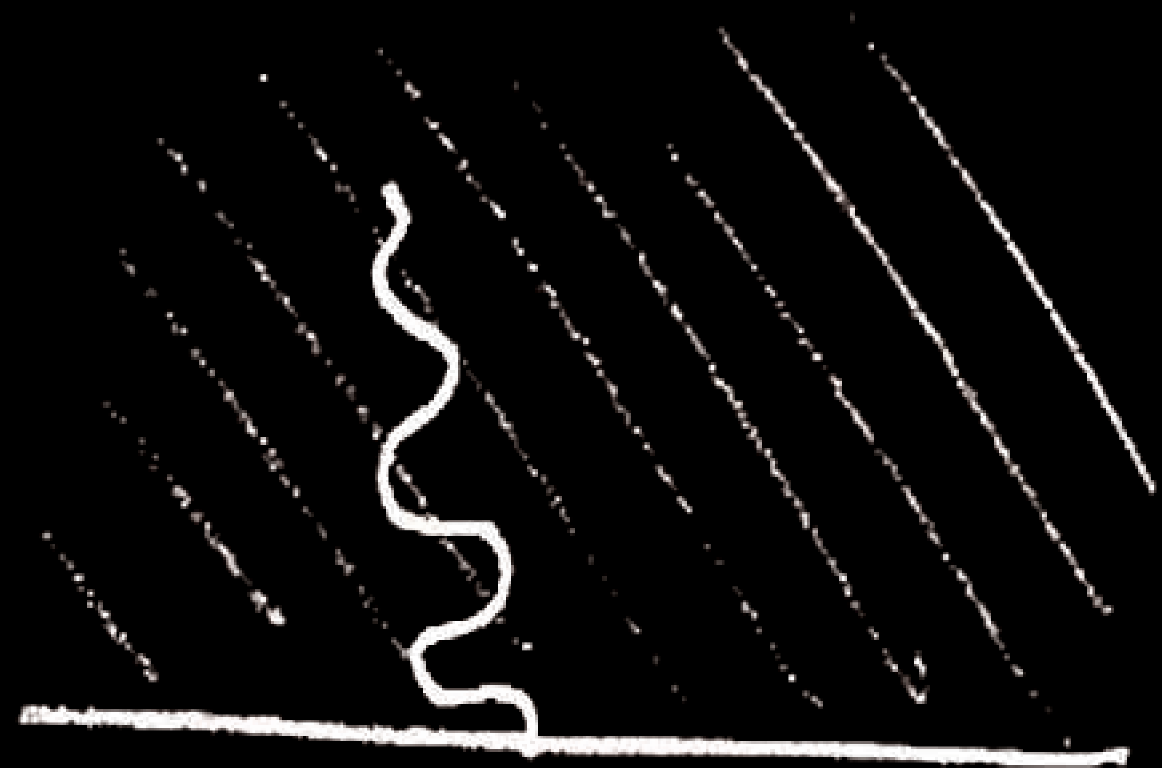
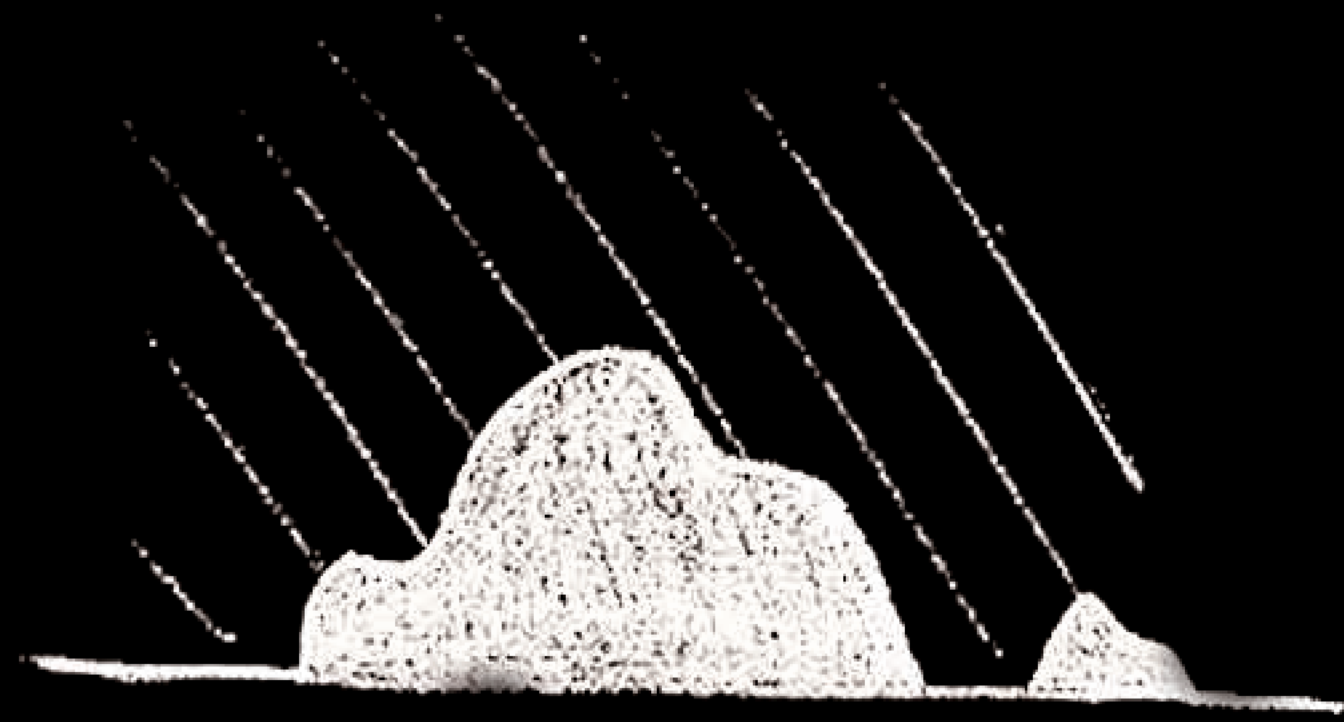
3. Half-plane capacity

A bounded subset K of the upper half-plane $\mathbb{H}$ is called a <u>compact</u> $\mathbb{H}$-<u>hull</u> if $K = \mathbb{H} \cap \overline{K}$ and $\mathbb{H} \setminus K$ is simply connected.

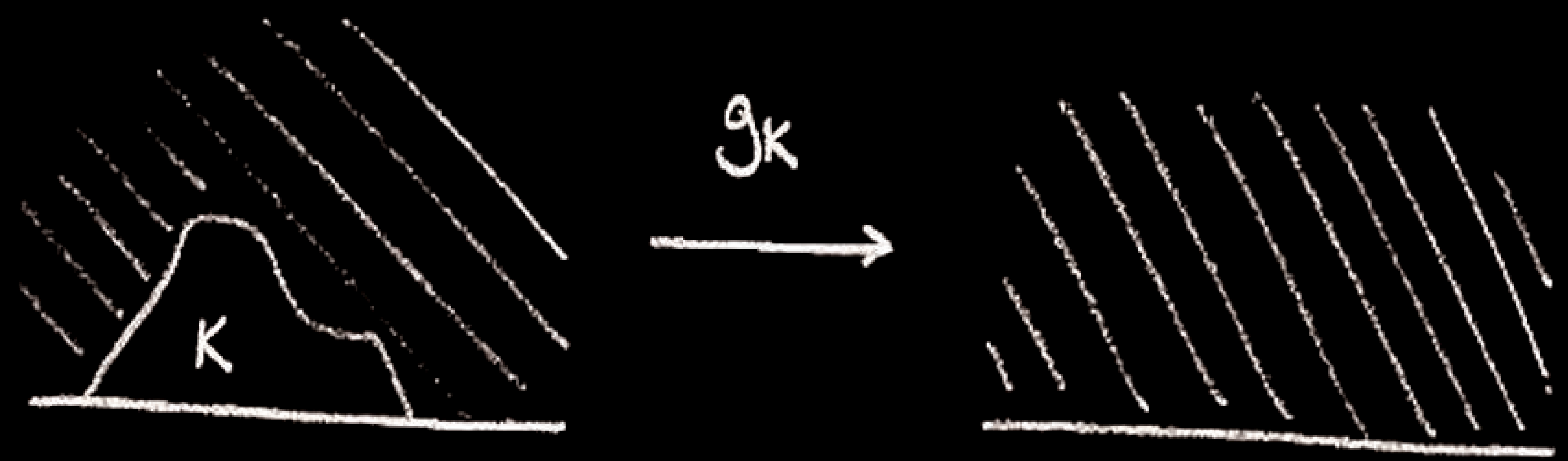




Proposition 3.1

Let K be a compact $\mathbb{H}$-hull. Set $H = \mathbb{H} \setminus K$.
There exists a unique conformal isomorphism
$g_K : H \to \mathbb{H}$ such that $g_K(z) - z \to 0$ as $z \to \infty$.
Moreover, for some $a_K \in \mathbb{R}$,

$$g_K(z) = z + \frac{a_K}{z} + O(|z|^{-2}), \quad z \to \infty.$$



Proposition 3.2

Let K be a compact $\mathbb{H}$-hull, $K \subseteq \mathbb{D}$

Set $a_K(\theta) = \mathbb{E}_{e^{i\theta}}(\mathrm{Im}(B_T))$, $\theta \in (0,\pi)$,
where B is a complex Brownian motion starting from $e^{i\theta}$, and $T = T(K) = \inf\{t \geq 0 : B_t \notin \mathbb{H} \setminus K\}$.

Then
$$a_K = \int_0^\pi a_K(\theta)\, p(\theta)\, d\theta \geq 0$$

where $p(\theta) = 2\sin\theta / \pi$. Moreover, there is a constant $C < \infty$, independent of K, such that

$$\left| g_K(z) - z - \frac{a_K}{z} \right| \leq \frac{C a_K}{|z|^2}, \quad |z| \geq 2.$$

Write $a_K = \mathrm{hcap}(K)$ half-plane capacity

Properties

- $\mathrm{hcap}(K) \geqslant 0$,
- $K \subseteq K' \Rightarrow \mathrm{hcap}(K) \leq \mathrm{hcap}(K')$ with equality only if $K = K'$,
- for $r > 0$, $\mathrm{hcap}(rK) = r^2\,\mathrm{hcap}(K)$,

  (indeed $g_{rK}(z) = r\,g_K(\frac{z}{r})$ by uniqueness in Proposition 3.1)
- for $b \in \mathbb{R}$, $\mathrm{hcap}(K+b) = \mathrm{hcap}(K)$,

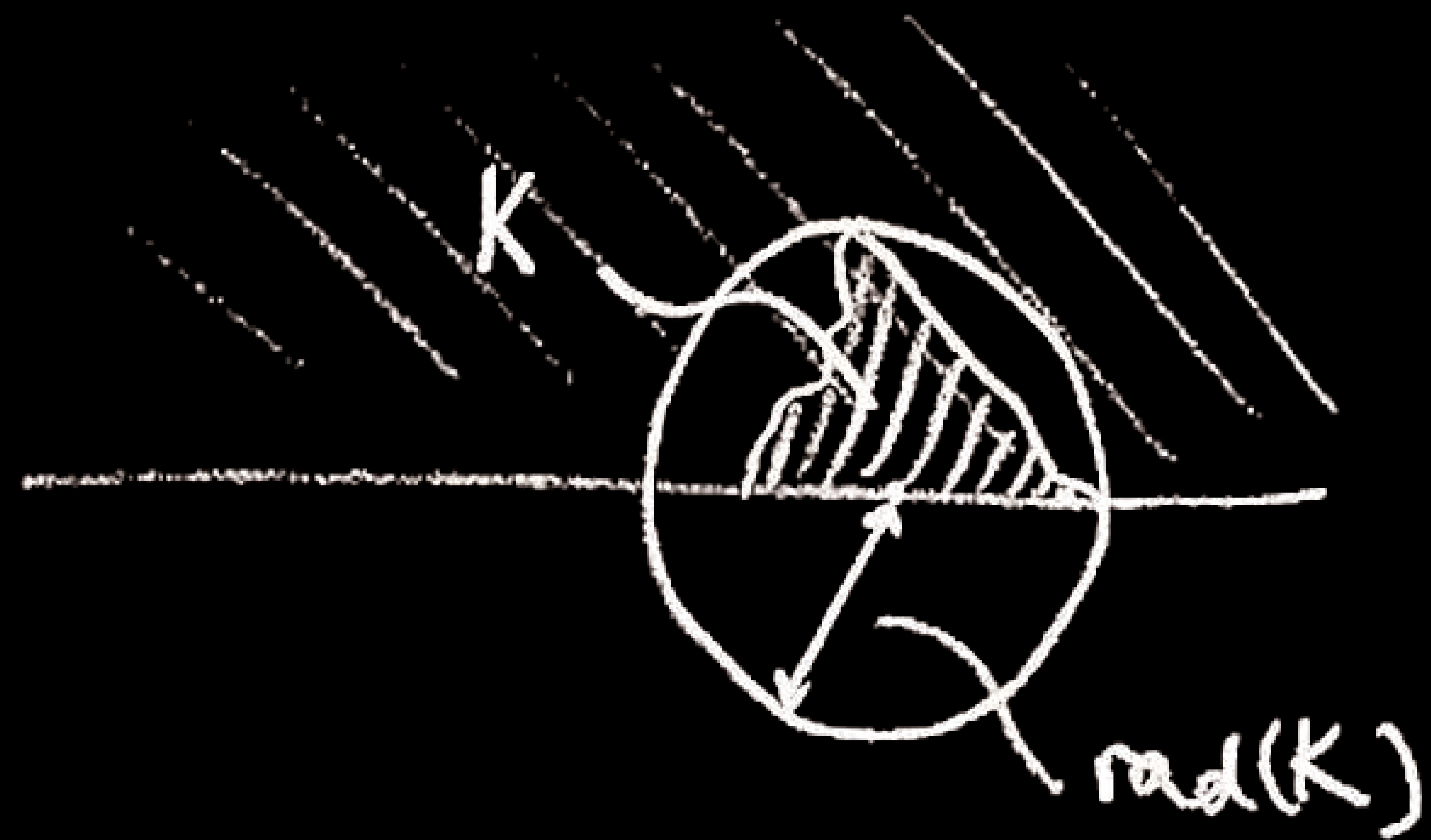
  (and $g_{K+b}(z) = g_K(z-b) + b$)

- for $S = (0, i]$, $\quad g_S(z) = \sqrt{z^2+1}$, $\quad \mathrm{hcap}(S) = \frac{1}{2}$

- for $A = \overline{\mathbb{D}} \cap \mathbb{H}$, $\quad g_A(z) = z + z^{-1}$, $\quad \mathrm{hcap}(A) = 1$

- $\mathrm{hcap}(K) \leq \mathrm{rad}(K)^2$ for all K

  where $\mathrm{rad}(K)$ is the radius of the smallest ball centred on the real axis which contains K



- there is a $C < \infty$ such that, for all K, for all z,

$$|z - g_K(z)| \leq C\,\mathrm{rad}(K),$$

- $$\mathrm{hcap}(K) = \lim_{y \to \infty} y\,\mathbb{E}_{iy}\left(\mathrm{Im}(B_T)\right).$$

Think of half-plane capacity as measuring the "average height of the boundary seen by a Brownian motion started at ∞".