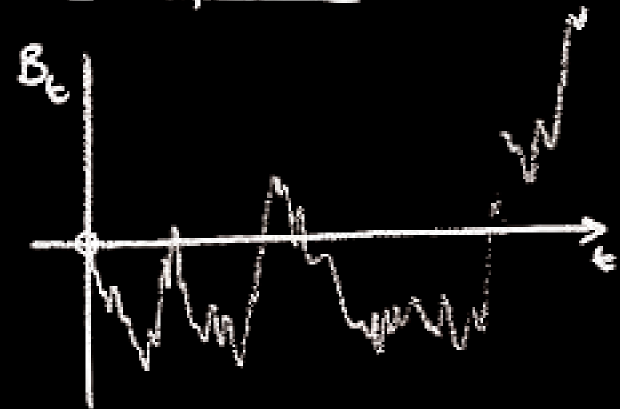
2. Brownian motion and conformal isomorphisms



A real-valued process $B = (B_t)_{t \geq 0}$ is a <u>Brownian motion</u> if:

(i) $B_0 = 0$,

(ii) $t \mapsto B_t(\omega) : [0, \infty) \to \mathbb{R}$
is continuous for all ω,

(iii) for all $s, t \geq 0$ with $s \leq t$, $B_t - B_s \sim N(0, t-s)$
and $B_t - B_s$ is independent of $\sigma(B_r : r \leq s)$.

Thus, for $s_1, \dots, s_n \geq 0$, and $t_k = s_1 + \dots + s_k$,
$(B_{t_1}, \dots, B_{t_n})$ has density function $\prod_{k=1}^{n} p(s_k, x_{k-1}, x_k)$
where $p(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-|x-y|^2/2t}$, $x_0 = 0$.

For Brownian motion starting from x, replace (i) by $B_0 = x$.

A <u>complex Brownian motion</u> starting from $z = x + iy$ is a complex-valued process $Z = (Z_t)_{t \geq 0}$ whose real and imaginary parts are independent Brownian motions starting from x, y respectively.

A <u>domain</u> in $\mathbb{C}$ (or $\mathbb{R}^n$) is a connected open set.

An <u>analytic function</u> f defined on a domain D in $\mathbb{C}$ is one which is complex differentiable.

For such a function $f = u + iv$, the Cauchy-Riemann equations hold

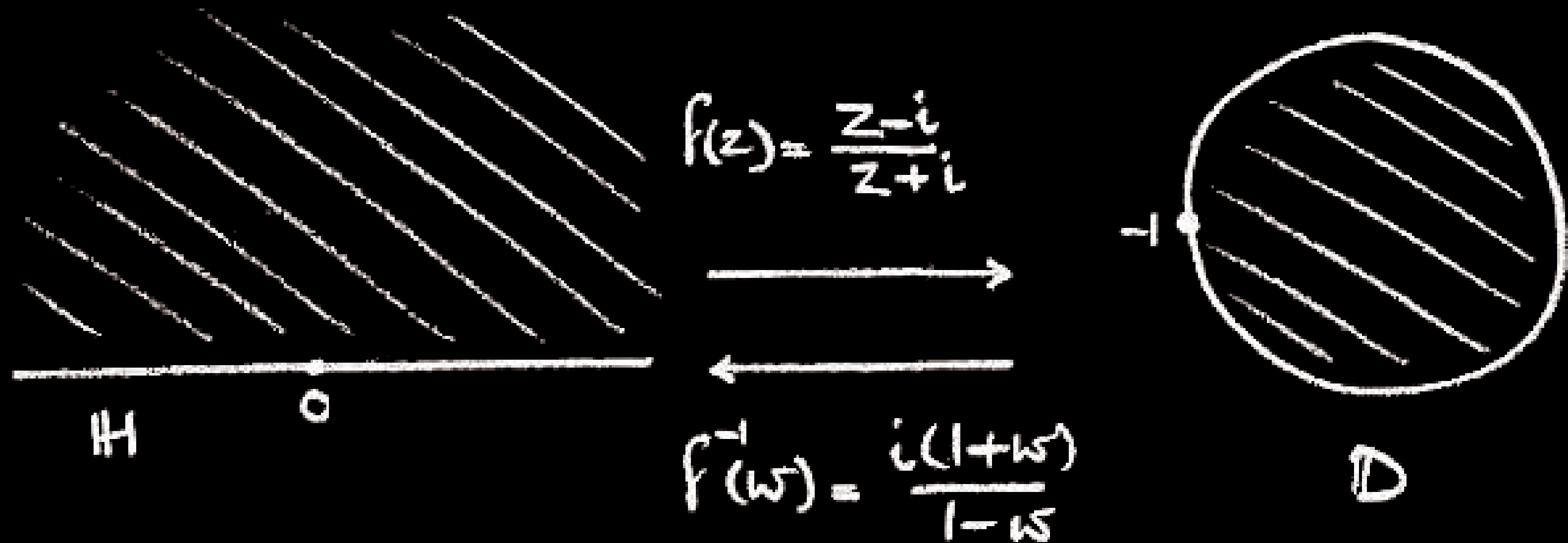
$$u_x = v_y \,, \quad u_y = -v_x$$

and hence u, v are harmonic in D

$$u_{xx} + u_{yy} = 0 \,, \qquad v_{xx} + v_{yy} = 0 \,.$$

We will call an analytic bijection $f : D \to D'$, for domains D, D' a <u>conformal isomorphism</u>.

For such f, the inverse function $f^{-1} : D' \to D$ is also analytic.



Itô's formula

Let u be a C^2 function on a domain D in $\mathbb{R}^n$.

Let Z be a continuous semimartingale with values in D.

Then, for all $t \geq 0$, a.s.

$$u(Z_t) = u(Z_0) + \underbrace{\int_0^t \partial_i u(Z_s)\, dZ_s^i}_{\text{Itô integral}} + \frac{1}{2}\int_0^t \partial_i \partial_j u(Z_s)\, \underbrace{dZ_s^i\, dZ_s^j}.$$

convenient notation for $d[Z^i, Z^j]$, where $[Z^i, Z^j]$ is the covariation process of Z^i, Z^j

For a complex Brownian motion $Z = X + iY = (X, Y)$,

$$dX_t\, dX_t = dY_t\, dY_t = dt, \quad dX_t\, dY_t = 0.$$

So, if $u : D \longrightarrow \mathbb{R}$ is harmonic,

$$\partial_i \partial_j u(Z_t)\, dZ_t^i\, dZ_t^j = (u_{xx} + u_{yy})(Z_t)\, dt = 0$$

and so

$$u(Z_t) = u(Z_0) + \int_0^t \nabla u(Z_s)\, dZ_s.$$

Itô integrals preserve local martingales so $u(Z)$ is a local martingale.

By the <u>optional stopping theorem</u>, if M is a local martingale and if T is a stopping time, such that $(M_t : t \leq T)$ is uniformly bounded, then

$$\mathbb{E}(M_T) = \mathbb{E}(M_0).$$

Hence, if u is bounded and harmonic on D, and is continuous on $\bar{D}$, and if Z is a complex Brownian motion starting from $z \in D$, then

$$f(z) = \mathbb{E}(f(Z_{T_D}))$$

where $T_D = \inf\{t > 0 : Z_t \in \partial D\}$.

Lemma 2.1 (**Schwarz lemma**)

Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be an analytic function, with $f(0)=0$.

Then $|f(z)| \le |z|$ for all z.

Moreover, if $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$,

then, for some $\theta \in \mathbb{R}$, $f(z) = e^{i\theta} z$ for all z.

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Corollary 2.2

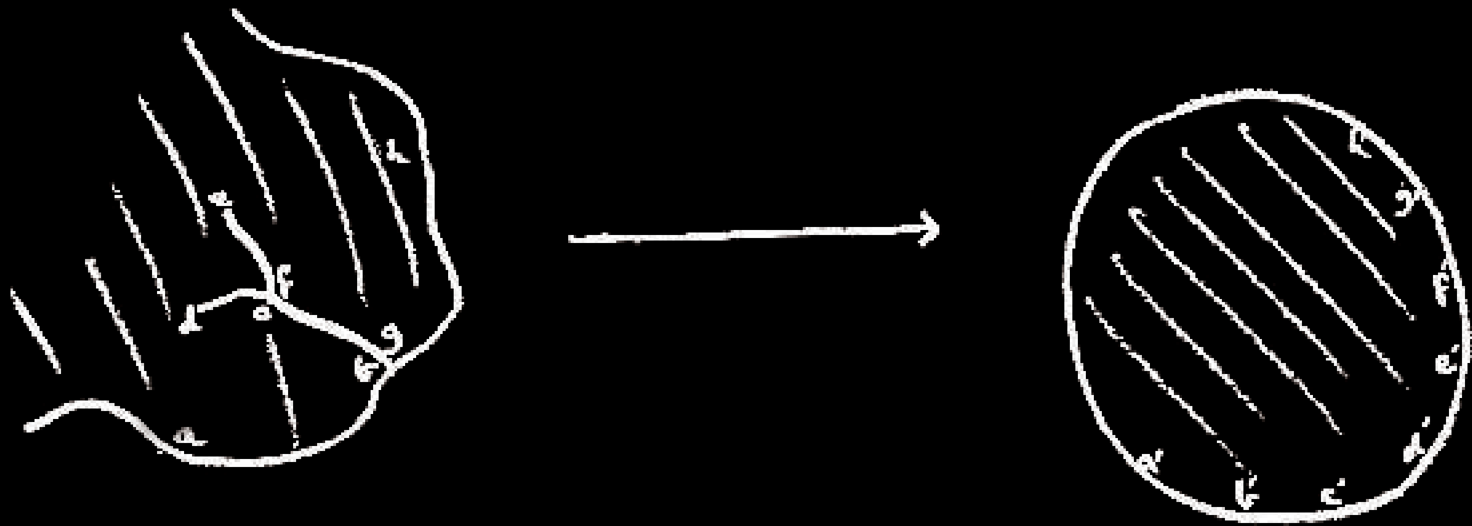
Let $\Phi : \mathbb{D} \to \mathbb{D}$ be a conformal isomorphism.

Then Φ is a <u>Möbius transformation</u>: there exists

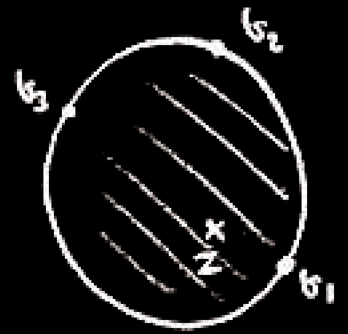
$w \in \mathbb{D}$ and $\theta \in \mathbb{R}$ such that $\Phi(z) = e^{i\theta} \dfrac{z - w}{1 - \bar{w} z}$ for all z.

Theorem 2.3 (*Riemann mapping theorem*)

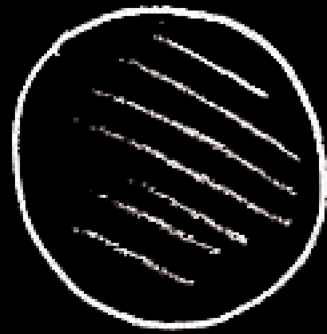
Let D be a simply connected proper domain in $\mathbb{C}$.
Then there exists a conformal isomorphism $\Phi : D \to \mathbb{D}$.



Möbius transformations on $\mathbb{D}$



$$\Phi(z) = e^{i\theta}\,\frac{z-w}{1-\bar{w}z}$$



For each of the following conditions, there exist $\theta \in [0, 2\pi)$ and $w \in \mathbb{D}$ such that Φ has the given property:

(i) $\Phi(z) = 0$ and $\Phi'(z) > 0$,

(ii) $\Phi(z) = 0$ and $\Phi(w_1) = 1$,

(iii) $\Phi(w_1) = 1$, $\Phi(w_2) = i$, $\Phi(w_3) = -1$.

In each case, θ and w are unique.

Theorem 2.4 (*Conformal invariance of Brownian motion*)

Let $\Phi : D \to D'$ be a conformal isomorphism.

Let $z \in D$, $z' \in D'$, with $z' = \Phi(z)$.

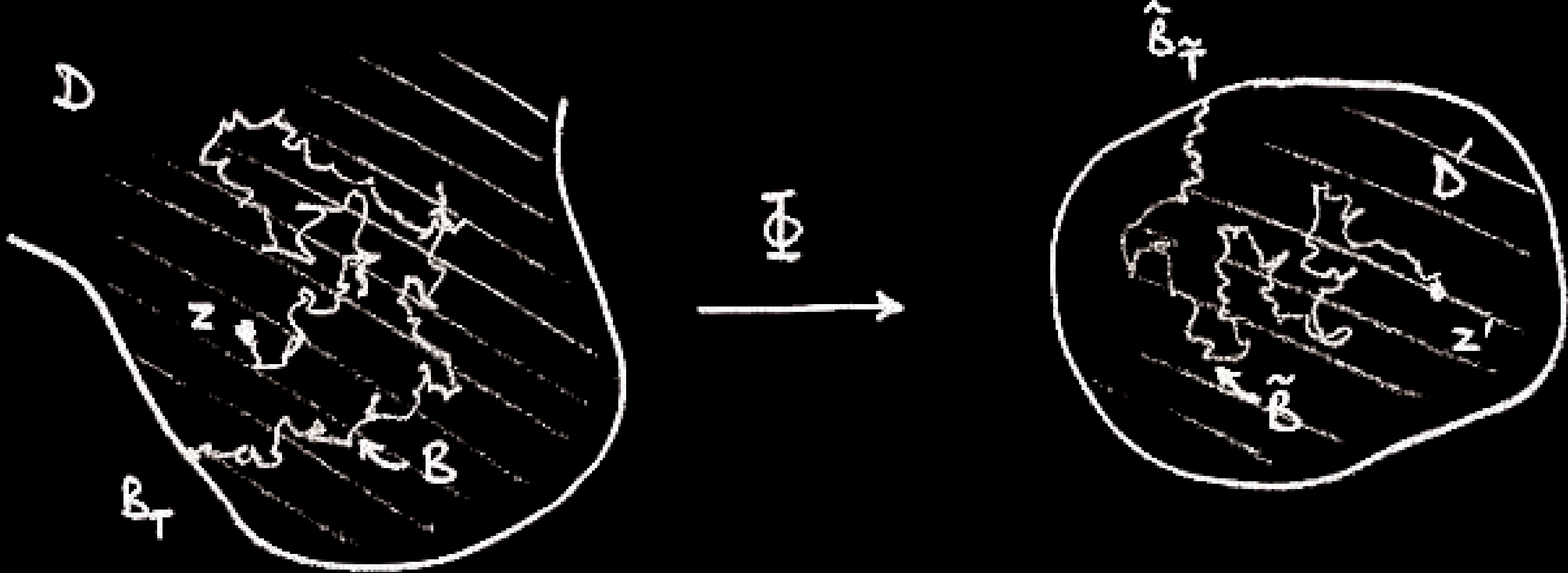
Suppose that B, B' are complex Brownian motions starting from z, z' respectively.

Set $T = \inf\{t \geq 0 : B_t \notin D\}$, $T' = \inf\{t \geq 0 : B'_t \notin D'\}$.

Define
$$A(t) = \int_0^t |\Phi'(B_s)|^2 ds, \quad t \leq T$$
and write τ for the inverse function $\tau = A^{-1} : [0, A(T)) \to [0, T)$.

Set $\tilde{T} = A(T)$, $\tilde{B}_t = \Phi(B_{\tau(t)})$.

Then $(\tilde{T}, (\tilde{B}_t)_{t<\tilde{T}})$ and $(T', (B'_t)_{t<T'})$ have the same distribution.



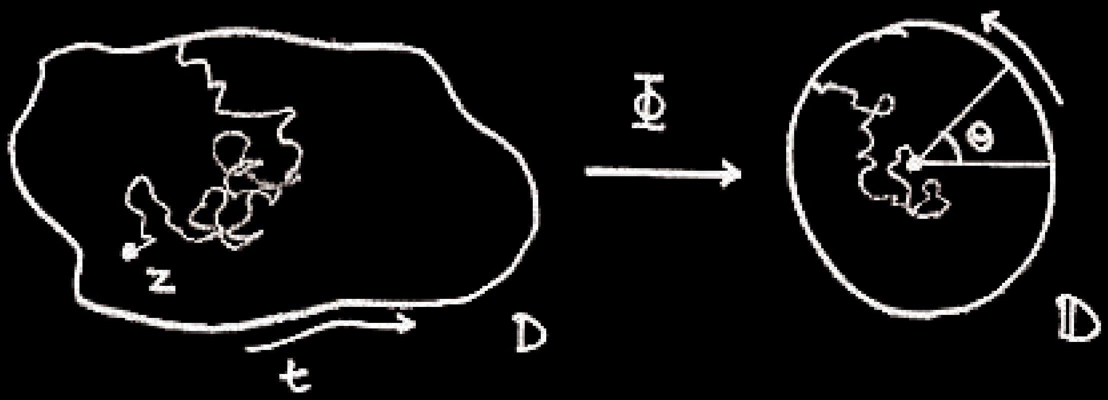
By considering the case $D = \mathbb{D}$, for example, B_t converges as $t \uparrow T$ to some $B_T \in \hat{D} \setminus D$.

So B defines a random chord in D, from z to $\hat{D} \setminus D$

Write $\mu_{D,z}$ for its distribution. Then $\mu_{D',z'} = \mu_{D,z} \circ \Phi^{-1}$.

Any Brownian chord has a natural parametrization by quadratic variation.

Calculation of hitting densities for Brownian motion



Parametrize the conformal boundary by t.

Then the hitting density is given by

$$h_D(z,t) = \frac{1}{2\pi} \frac{d\theta}{dt}$$

$h_{\mathbb{D}}(0,\theta)$

Hence, for each $z \in \mathbb{D}$,

$$h_{\mathbb{D}}(z,t) = \frac{1}{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2}, \qquad 0 \le t < 2\pi.$$

Also, for $z = x+iy \in \mathbb{H}$,

$$h_{\mathbb{H}}(z,t) = \frac{1}{\pi} \operatorname{Im}\left(\frac{1}{t-z}\right) = \frac{y}{\pi((x-t)^2+y^2)}, \qquad t \in \mathbb{R}.$$

The measure $h_D(z,t)dt$ is called <u>harmonic measure</u>.

We have

$$u(z) = \int u(t) h_D(z,t) dt$$

for any function u harmonic in D and continuous on $\hat{D}$.

Lemma 2.5

Let u be harmonic in a domain D. Then, for all $z \in D$,

$$|u_x(z)| \leq \frac{4 \|u\|_\infty}{\pi \operatorname{dist}(z, \partial D)}$$