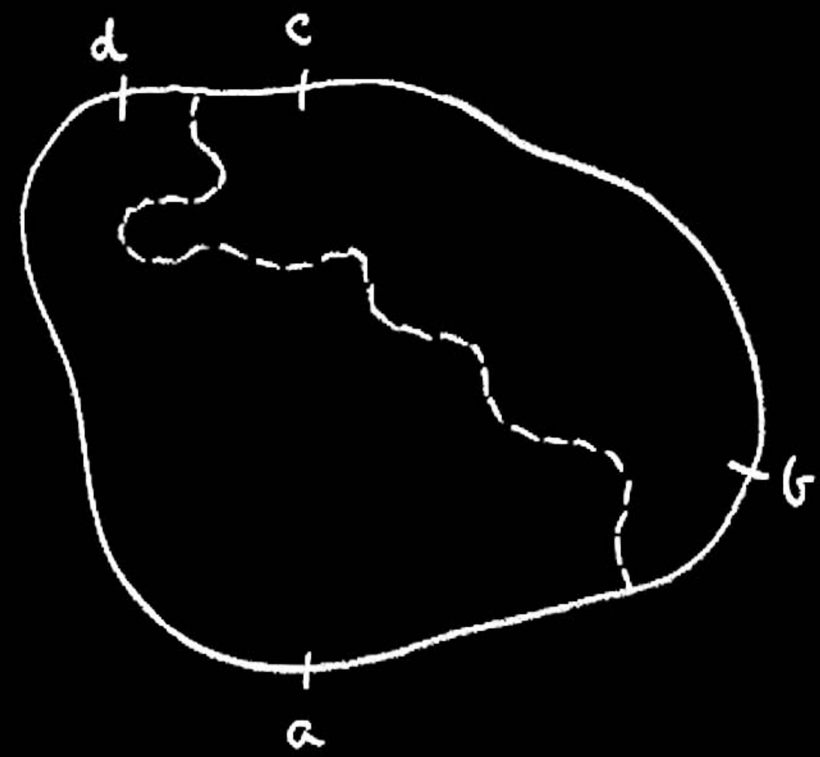
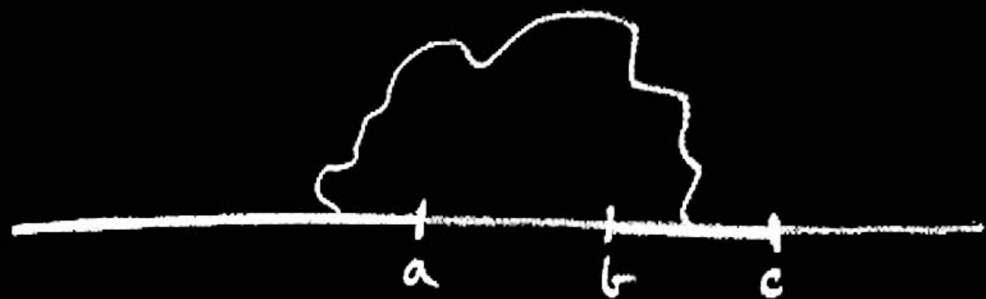
10. Cardy's Formula

Consider the hexagon percolation model



Is there a path of white hexagons from [a, b] to [c, d] ?

Assuming conformal invariance, reduce to $\mathbb{H}$



Is there a path of white hexagons from $(-\infty, a]$ to $[b, c]$?

Two ways to explore this question:

1. Place black hexagons on $(-\infty, a]$, white on $[a, \infty)$ and see if black/white interface from a hits $[b, c]$

Cardy obtained the value $\phi\left(\frac{c-b}{c-a}\right)$ for the crossing probability.

In the continuum limit, by Proposition 6.4, for SLE(6)

$$\mathbb{P}(\gamma \text{ hits } [x,y]) = \phi\left(\frac{y-x}{y}\right)$$

which is consistent with Cardy's formula $(a=0, b=x, c=y)$

Here

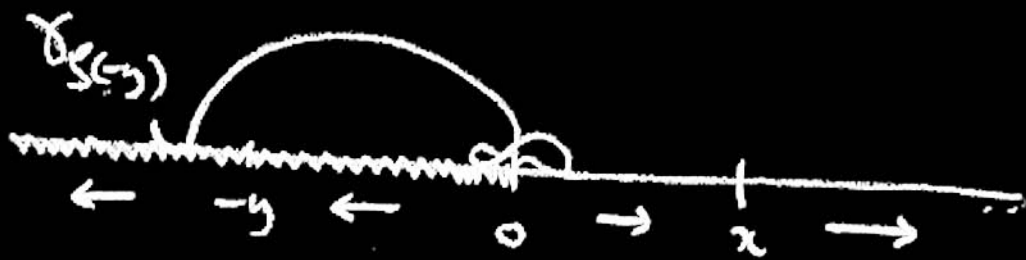
$$\phi(\theta) \propto \int_0^{\theta} \frac{du}{u^{2-4a}(1-u)^{2a}} \quad , \quad \theta \in [0,1], \ \phi(1)=1$$

$$= \int_0^{\theta} \frac{du}{u^{2/3}(1-u)^{2/3}} \quad \text{since} \quad a = \frac{2}{K} = \frac{1}{3}$$

2. Place black hexagons on $(-\infty, b]$, white on $[b, \infty)$ and see if black/white interface from b hits $(-\infty, a]$ before $[c, \infty)$.

We should therefore find for SLE(6) and $x, y > 0$

$$\mathbb{P}(\zeta(-y) < \zeta(x)) = \phi\left(\frac{x}{x+y}\right)$$



Proposition 10.1

Let γ be an SLE(κ), with $\kappa = \frac{2}{a} \in (4, \infty)$. Take $x, y > 0$.

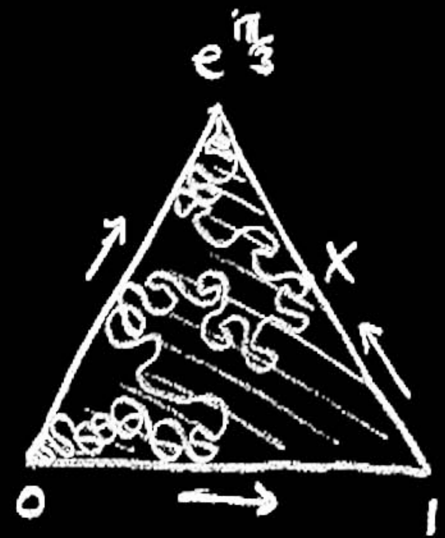
Then $$\mathbb{P}(S(-y) < S(x)) = \psi\left(\frac{x}{x+y}\right)$$

where $$\psi(\theta) \propto \int_0^{\theta} \frac{du}{u^{2a}(1-u)^{2a}}, \quad \theta \in [0,1], \quad \psi(1)$$

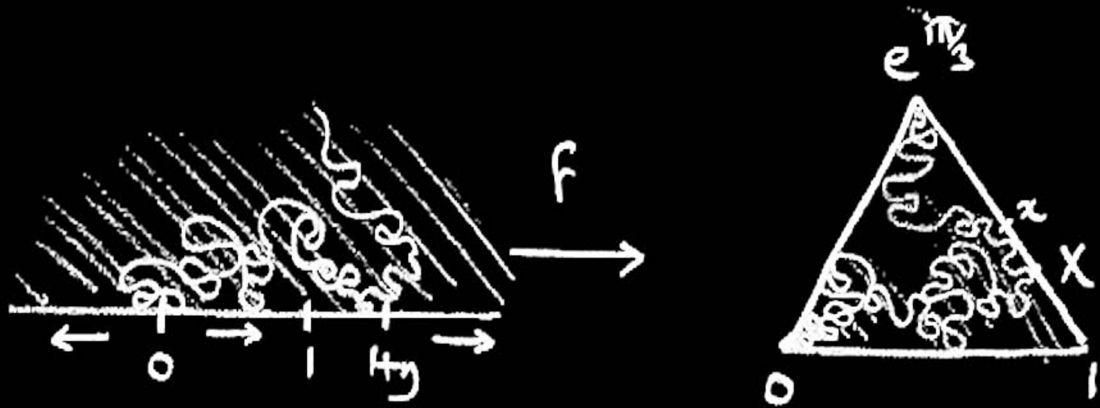
Note that, if $\kappa = 6$, then $\psi = \phi$.

Carleson's observation

Let γ be SLE(6) in $(\Delta, 0, e^{i\pi/3})$.
Then the point X at which γ hits $[1, e^{i\pi/3}]$ is uniformly distributed on that interval.



Proof



The Schwarz-Christoffel transformation $(\mathbb{H},0,1,\infty) \to (\Delta,0,1,e^{i\pi/3})$ is

$$f(z) = c\int_0^z \frac{dw}{w^{2/3}(1-w)^{2/3}}, \quad c = \Gamma(2/3)\Big/\Gamma(1/3)^2.$$

In particular

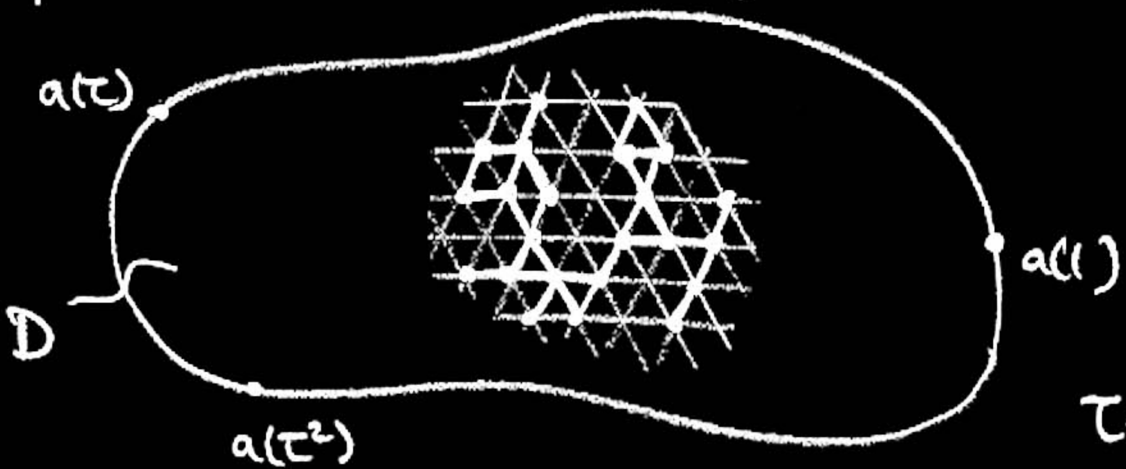
$$f(1+y) = 1 + e^{2\pi i/3}x, \quad x = \phi\left(\frac{y}{1+y}\right), \quad 0<y<\infty.$$

So, by conformal invariance and Cardy's formula

$$\mathbb{P}(X \le x) = \mathbb{P}(\text{SLE}(6) \text{ in } (\mathbb{H},0,\infty) \text{ hits } [1,1+y]) = \phi\left(\frac{y}{1+y}\right) = x. \quad \square$$

Smirnov's theorem

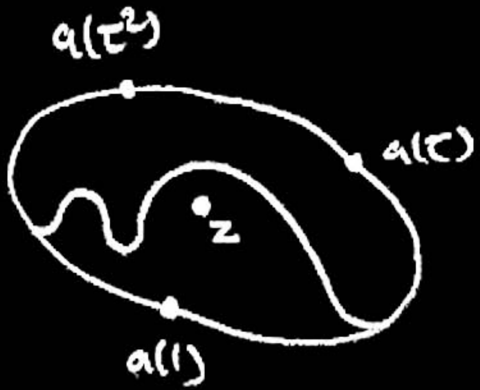
Consider the hexagon percolation model, thought of as site percolation on the triangular lattice of edge length δ.



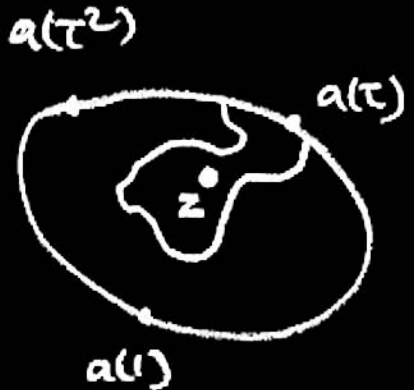
$\tau = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}\,i}{2}$

For $z \in D$ and $\alpha \in \{1, \tau, \tau^2\}$, write $Q_\alpha(z)$ for the event that z is separated from $a(\tau\alpha)a(\tau^2\alpha)$ by a black path from $a(\alpha)a(\tau\alpha)$ to $a(\tau^2\alpha)a(\alpha)$

Set $H_\alpha^\delta(z) = \mathbb{P}(Q_\alpha(z))$



$Q_1(z)$



$Q_\tau(z)$

Let h'_α be the unique affine function on $\underset{a'(1)\quad a'(\tau)}{\overset{a'(\tau^2)}{\Delta}}$

such that $h'_\alpha(a'(\alpha)) = 1$, $h'_\alpha(z) = 0$ on $a(\tau\alpha)a(\tau^2\alpha)$

Set $h_\alpha = h'_\alpha \circ \Phi$ where Φ is the conformal isomorphism

$$(D, a(1), a(\tau), a(\tau^2)) \longrightarrow (\Delta, a'(1), a'(\tau), a'(\tau^2))$$

Theorem 10.2

For $\alpha = 1, \tau, \tau^2$, H_α^δ converges to h_α uniformly on D as $\delta \downarrow 0$.

In particular, taking $z \in \partial D$, this implies asymptotic conformal invariance of crossing probabilities and Cardy's formula in the case of the triangular lattice.

Harmonic triples and $2\pi/3$-Cauchy-Riemann equations

For $\alpha = 1, \tau, \tau^2$, $\tau = e^{2\pi i/3}$, and f analytic, set

$$f_\alpha = \mathrm{Re}\left(\frac{f}{\alpha}\right).$$

Then f_α is harmonic and we can recover f by

$$\alpha f = f_\alpha + \frac{i}{\sqrt{3}}\left(f_{\alpha\tau} - f_{\alpha\tau^2}\right).$$

Also, for any $\eta \in \mathbb{C}$, the directional derivatives satisfy

$$\nabla_\eta f_\alpha(z) := \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} \mathrm{Re}\left(\frac{f(z+\varepsilon\eta)}{\alpha}\right)$$

$$= \mathrm{Re}\left(\frac{f'(z)\eta}{\alpha}\right) = \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} \mathrm{Re}\left(\frac{f(z+\varepsilon\tau\eta)}{\alpha\tau}\right) = \nabla_{\tau\eta} f_{\alpha\tau}(z)$$

These are the $2\pi/3$-<u>Cauchy-Riemann equations</u> and $(f_1, f_\tau, f_{\tau^2})$ is the <u>harmonic triple</u> of f.

Conversely, if we are given C^1 functions f_1, f_τ, f_{τ^2} such that, for $\alpha \in \{1, \tau, \tau^2\}$, for all η

$$\nabla_\eta f_\alpha = \nabla_{\tau\eta} f_{\tau\alpha},$$

then f, defined by

$$f = f_1 + \frac{i}{\sqrt{3}}(f_\tau - f_{\tau^2})$$

is analytic and $f_\alpha = \mathrm{Re}(\frac{f}{\alpha})$ for all α.

(To show f is analytic we check that its integral vanishes round all triangles $(z, z+l, z+l+l\tau)$, $l>0$.)

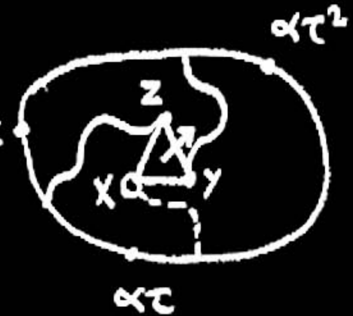
Sketch proof of 10.2

$$H_\alpha(z+\eta) - H_\alpha(z) = P_\alpha(z,\eta) - P_\alpha(z+\eta,-\eta)$$

where

$$P_\alpha(z,\eta) = \mathbb{P}(Q),$$

$$Q = Q_\alpha(z+\eta)\setminus Q_\alpha(z) = \Big\{$$



$$\Big\}$$

But

$$\mathbb{P}(Q) = \mathbb{P}\Big($$

$$\Big) = \mathbb{P}\big(Q_{\tau\alpha}(z+\tau\eta)\setminus Q_{\tau\alpha}(z)\big)$$

So $\boxed{P_\alpha(z,\eta) = P_{\tau\alpha}(z,\tau\alpha)}$

Lemma 10.3 (Hölder estimate)

There are constants $\varepsilon > 0$, $C < \infty$, depending only on $(D, a(1), a(\tau), a(\tau^2))$ such that

$$|H_\alpha(z) - H_\alpha(z')| \leq C(|z-z'| \vee \delta)^\varepsilon.$$

Also, $H_\alpha(a(\alpha)) \to 1$ as $\delta \downarrow 0$.

Consider the discrete contour integrals

$$\int_\Delta^\delta H(z)\,dz = \delta \sum_{z\in A} H(z) + \delta\tau \sum_{z\in B} H(z) + \delta\tau^2 \sum_{z\in C} H(z)$$

Lemma 10.4

$$\int_\Delta^\delta H_\alpha(z)\,dz = \frac{1}{\tau}\int_\Delta^\delta H_{\alpha\tau}(z)\,dz + O(\ell\delta^\varepsilon)$$

Proof: Resummation using $P_\alpha(z,\eta) = P_{\tau\alpha}(z,\tau\eta)$ together with $P_\alpha(z,\eta) \le C\delta^\varepsilon$ from Hölder estimate for stray terms.

By Lemma 10.3, every sequence $\delta_n \downarrow 0$ contains a subsequence δ_{n_k} such that $H_\alpha^{\delta_{n_k}}$ converges uniformly on D.

By Lemma 10.4, for any such subsequence, the limits, f_α say satisfy $f_\alpha(a(\alpha)) = 1$, $f_\alpha(z) = 0$ on $a(\alpha\tau)a(\alpha\tau^2)$ and

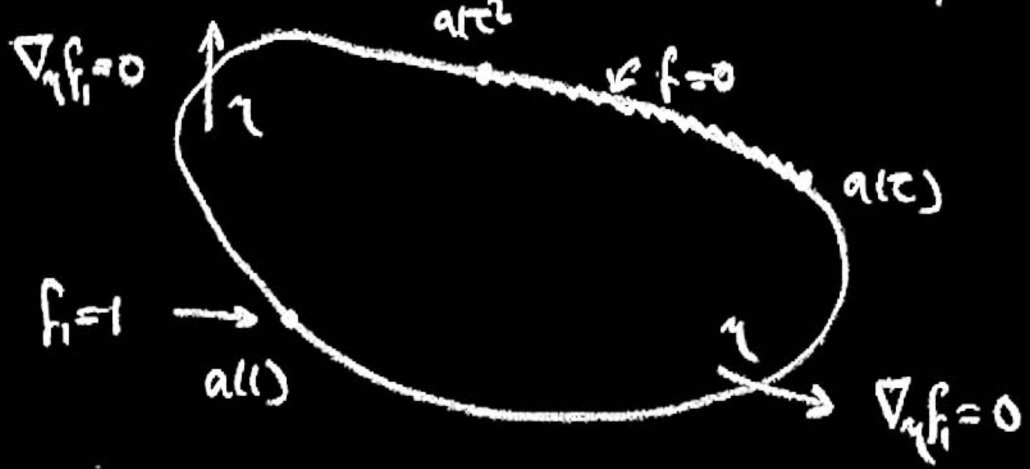
$$\int_\Delta f_\alpha(z)\,dz = \frac{1}{\tau}\int_\Delta f_{\alpha\tau}(z)\,dz$$

Form $f = f_1 + \frac{i}{\sqrt{3}}(f_\tau - f_{\tau^2})$ then f is analytic by Morera's theorem, and $f_\alpha = \mathrm{Re}(F_\alpha)$.

Hence

$$\nabla_\eta f_\alpha = \nabla_{\tau\eta} f_{\tau\alpha} \qquad (\text{limiting form of } P_\alpha(z,\eta) = P_{\tau\alpha}(z,\tau\alpha))$$

so f_1 satisfies the Dirichlet-Neumann problem



which has h_1 as its unique solution