1. Introduction

- Random chords and fillings
- Conformal invariance
- Restriction, locality, and domain Markov properties
- Examples: Brownian chord, self-avoiding walk, percolation, loop-erased walk

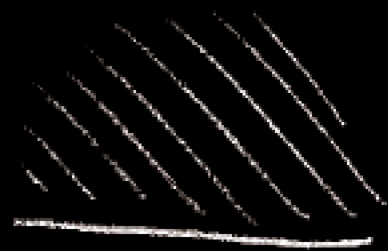
U simply connected, $\underbrace{\text{proper}}_{\text{not } \emptyset \text{ or } \mathbb{C}}$ $\underbrace{\text{domain}}_{\text{open and connected}}$ in $\mathbb{C}$

By the Riemann mapping theorem, there exists an analytic bijection Φ from U to the open unit disk $\mathbb{D}$. Write $\hat{U}$ for the completion of U in the metric

$$d(z,w) = |\Phi(z) - \Phi(w)|.$$

Jordan domain

$\hat{U} = \overline{U}$

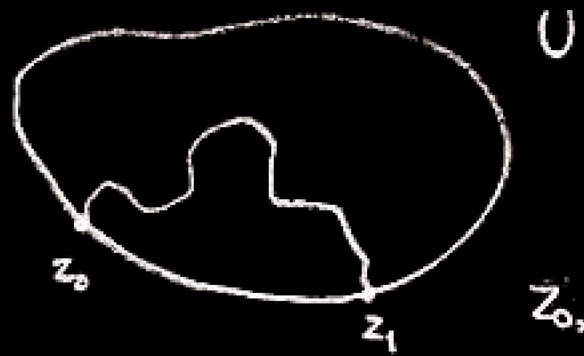


upper half-plane

$\hat{\mathbb{H}} = \overline{\mathbb{H}} \cup \{\infty\}$

$z_1 = z_2$ in $\overline{U}$

$z_1 \neq z_2$ in $\hat{U}$



$z_0, z_1 \in \hat{U}$ $\qquad D = (U, z_0, z_1)$

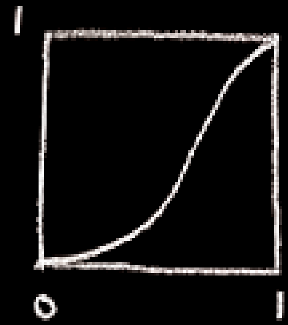
A <u>path</u> in D is a continuous map $\gamma : [0,1] \to \hat{U}$ with $\gamma(0) = z_0$, $\gamma(1) = z_1$.

Define a filtration $(\mathcal{P}_{D,t})_{0 \le t \le 1}$ and σ-algebra $\mathcal{P}_D$ on the set P_D of paths in D by

$$\mathcal{P}_{D,t} = \sigma(\gamma \mapsto \gamma_s : s \le t), \qquad \mathcal{P}_D = \mathcal{P}_{D,1}.$$

A <u>chord</u> in D is an equivalence class of paths under reparametrization:

$\gamma \sim \gamma' \iff \gamma' = \gamma \circ \varphi$ for some increasing homeomorphism φ of $[0,1]$.



Define a σ-algebra $\mathcal{C}_D$ on the set C_D of chords in D by

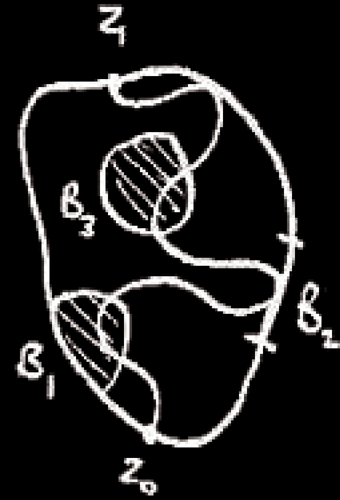
$$\mathcal{C}_D = \{A \subseteq C_D : \{\gamma : [\gamma] \in A\} \in \mathcal{P}_D\}.$$

Write $C = C_{(\mathbb{H},0,\infty)}$, $\mathcal{C} = \mathcal{C}_{(\mathbb{H},0,\infty)}$.

For paths γ, γ' in $D = (U, z_0, z_1)$ with $\gamma \sim \gamma'$ and for Borel sets $B_1, \dots, B_n \subseteq \hat{U}$,

$$\gamma_{t_1} \in B_1, \dots, \gamma_{t_n} \in B_n \text{ for some } t_1 < \dots < t_n$$

$$\Leftrightarrow \gamma'_{s_1} \in B_1, \dots, \gamma'_{s_n} \in B_n \text{ for some } s_1 < \dots < s_n.$$

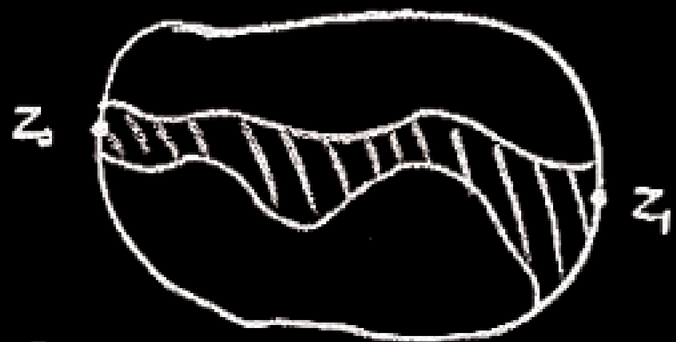


So define

$$A(B_1, \dots, B_n) = \{[\gamma] : \gamma_{t_1} \in B_1, \dots, \gamma_{t_n} \in B_n \text{ for some } t_1 < \dots < t_n\} \subseteq C_D$$

Then $A(B_1, \dots, B_n) \in \mathcal{C}_D$ and $\mathcal{C}_D$ is generated by sets of this form.

A *filling* in D is a closed, connected, simply connected subset of $\hat{U}$ containing z_0, z_1.



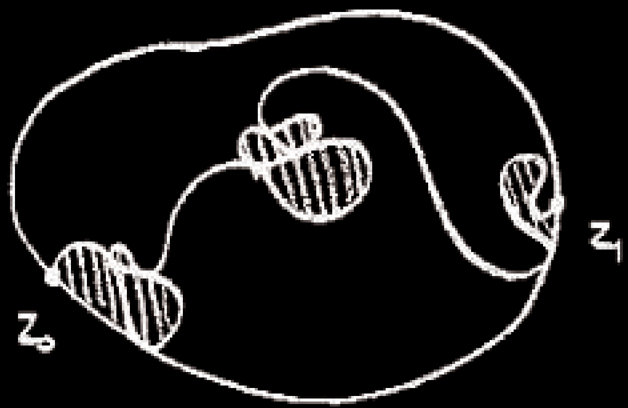
Write S_D for the set of such fillings.

Write $D' \subseteq D$ if

$U' \subseteq U$, $U' = U$ near z_0, z_1, $z_0 = z_0'$, $z_1 = z_1'$.

The family of sets $S_{D'}$, with $D' \subseteq D$ is a π-system on S_D (i.e. closed under finite intersections)

Write $\mathcal{F}_D$ for the σ-algebra generated by this π-system.



Can define $F : C_D \longrightarrow S_D$ by "filling in the chord".

Then F is $\mathcal{C}_D / \mathcal{S}_D$ measurable.

Write $S = S_{(\mathbb{H},0,\infty)}$, $\mathcal{S} = \mathcal{S}_{(\mathbb{H},0,\infty)}$.

Write $\mathcal{D}$ for the set of triples $D = (U, z_0, z_1)$.

For each $D \in \mathcal{D}$ we have defined

a measurable space of chords $(C_D, \mathcal{C}_D)$

a measurable space of fillings $(S_D, \mathcal{S}_D)$.

We are going to consider families of probability measures $(\mu_D : D \in \mathcal{D})$, either on chords or fillings.

Take $D = (U, z_0, z_1)$, $D' = (U', z_0', z_1')$ as above.

By the Riemann mapping theorem, there is a homeomorphism $\Phi : \hat{U} \longrightarrow \hat{U}'$, analytic on U (with inverse analytic on U'), such that
$\Phi(z_0) = z_0'$, $\Phi(z_1) = z_1'$.

Call such Φ a <u>conformal isomorphism</u> $D \to D'$.

The conformal automorphisms of $(\mathbb{H}, 0, \infty)$ are the <u>scalings</u> $\Phi_\sigma(z) = \sigma z$, $\sigma \in (0, \infty)$.

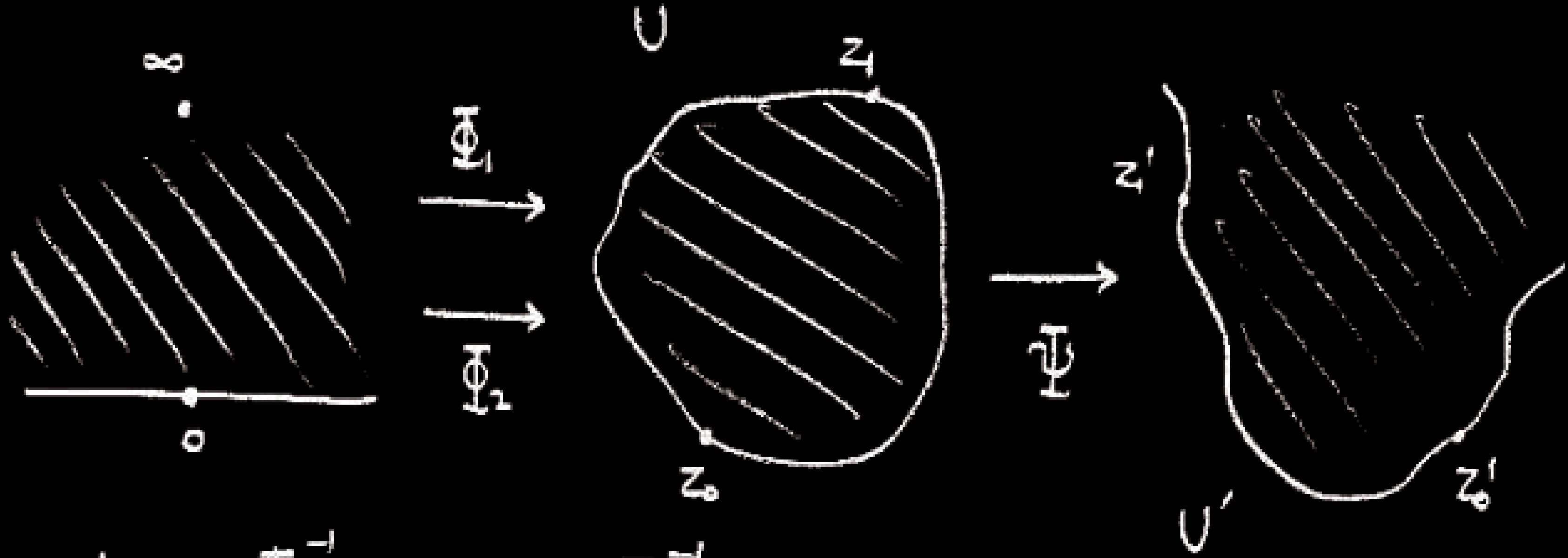
A measure μ on $(C, \mathcal{C})$ is <u>scale-invariant</u> if $\mu = \mu \circ \Phi_\sigma^{-1}$ for all scalings Φ_σ.

Given such a μ, we can define a family of measures μ_D on $(C_D, \mathcal{C}_D)$ for each $D \in \mathcal{D}$, by $\mu_D = \mu \circ \Phi_D^{-1}$, where $\Phi_D : (\mathbb{H}, 0, \infty) \to D$ is a conformal isomorphism.

Then $(\mu_D : D \in \mathcal{D})$ is <u>conformally invariant</u>:

$$\mu_{D'} = \mu_D \circ \Psi^{-1}$$

for any conformal isomorphism $\Psi : D \to D'$.



$\mu_1 = \mu \circ \Phi_1^{-1}$, $\mu_2 = \mu \circ \Phi_2^{-1}$

Note that $\Phi = \Phi_2^{-1} \circ \Phi_1$ is a scaling, so $\mu = \mu \circ \Phi^{-1}$.

Hence $\mu_2 = (\mu \circ \Phi^{-1}) \circ \Phi_1^{-1} = \mu_1$.

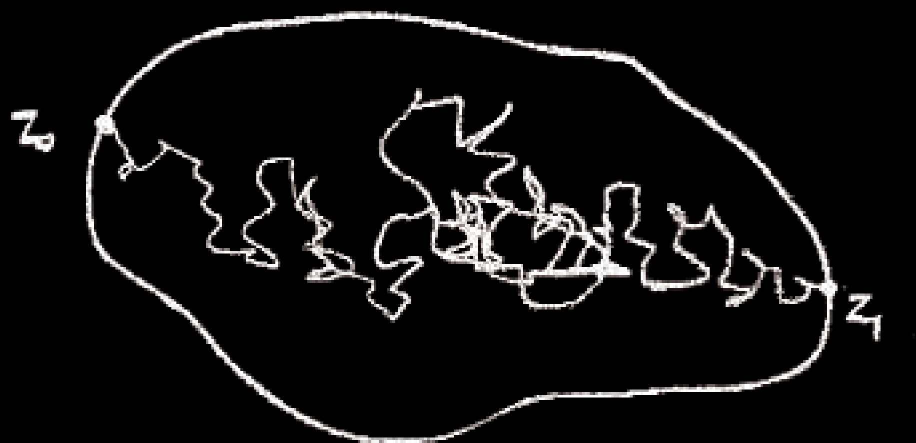
So μ_D is well-defined, and

$$\mu_{D'} = \mu \circ (\Psi \circ \Phi_1)^{-1} = \mu_D \circ \Psi^{-1}.$$

Example : Brownian chord

"Take a planar Brownian motion B, starting from z_0, run until it first leaves U, at time T_U say, and condition on $B_{T_U} = z_1$. So obtain a random chord $X = [(B_t : 0 \le t \le T_U)]$. Write μ_D for the law of X."

This can be made sense of by a limit argument.



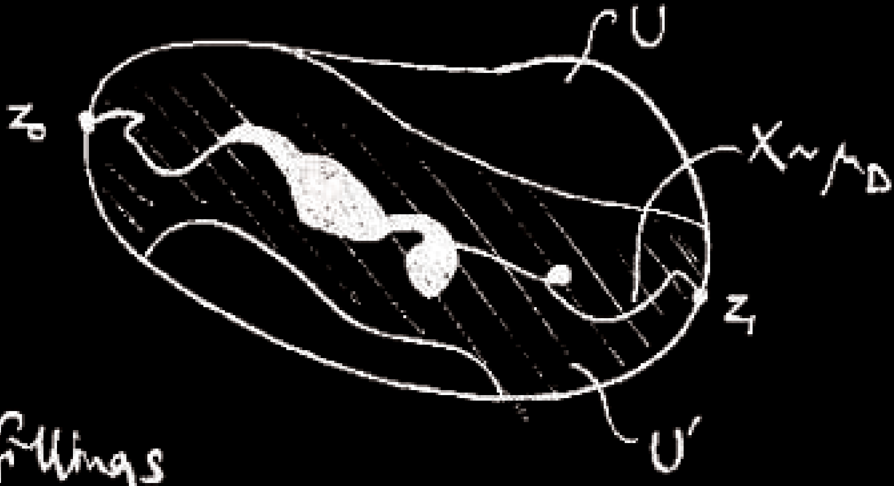
In fact $(\mu_D : D \in \mathcal{D})$ is conformally invariant, and $\mu_{(\mathbb{H}, 0, \infty)}$ can be constructed as follows.

Take B, W Brownian motions in $\mathbb{R}, \mathbb{R}^3$ respectively, starting from 0. We know that $R_t = |W_t| > 0$ for all $t > 0$ and $R_t \to \infty$ as $t \to \infty$.

So obtain $X = [((B_t, R_t) : 0 \leq t \leq \infty)] \in C$.

Then $\mu_{(\mathbb{H}, 0, \infty)}$ is the law of X.

Restriction property

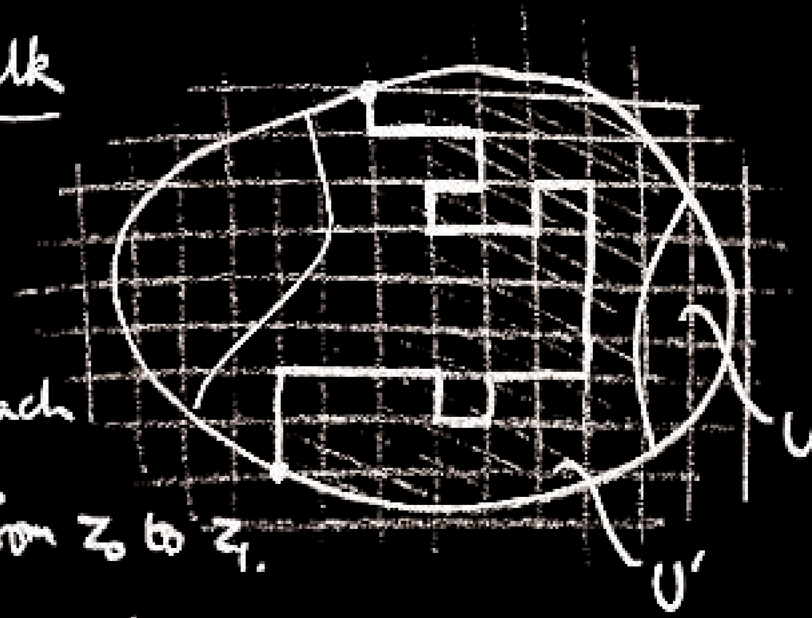


$(\mu_D : D \in \mathcal{D})$ a family of probability measures on fillings

Say $(\mu_D : D \in \mathcal{D})$ has the restriction property if
for all $D, D' \in \mathcal{D}$ with $D' \subseteq D$,
the conditional law of a random filling $X \sim \mu_D$,
given that $X \subseteq D'$ is $\mu_{D'}$.

(Think about uniform distributions
— also memoryless property of exponentials.)

Example : self-avoiding walk

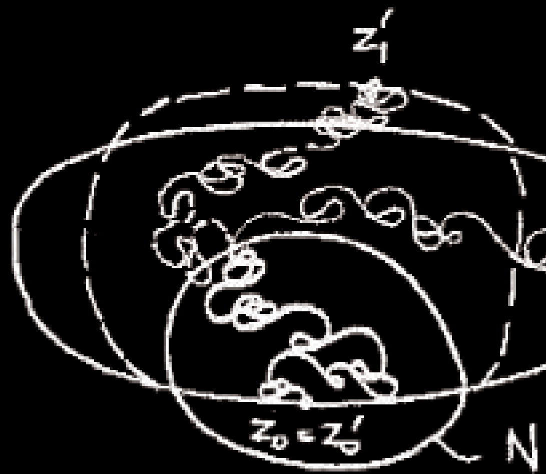


Fix $\alpha \in (0,1)$. Consider the probability measure μ_D^ε which assigns a mass $\propto \alpha^{-n}$ to each simple lattice path in $D \cap \varepsilon\mathbb{Z}^2$ from z_0 to z_1.

Then $(\mu_D^\varepsilon : D \in \mathcal{D})$ has the restriction property.

It is conjectured that, for a critical value of α, $\mu_D^\varepsilon \to \mu_D$ as $\varepsilon \downarrow 0$, and that $(\mu_D : D \in \mathcal{D})$ is conformally invariant. We later identify the only possible such limit using the restriction property.

Locality property



initial domain

$(\mu_D : D \in \mathcal{D})$
family of probability measures on chords

Say $(\mu_D : D \in \mathcal{D})$ has the locality property if
for all $D, D' \in \mathcal{D}$, for all 'initial domains N common to D, D',
if $X \sim \mu_D$, $X' \sim \mu_{D'}$, then $X^N \sim X'^N$, where X^N is X
stopped on leaving N

Example : percolation

Consider a honeycomb lattice of scale ε.

Colour each cell independently black or white, with probability $\frac{1}{2}$.

Trace the black/white interface from z_0 to z_1,

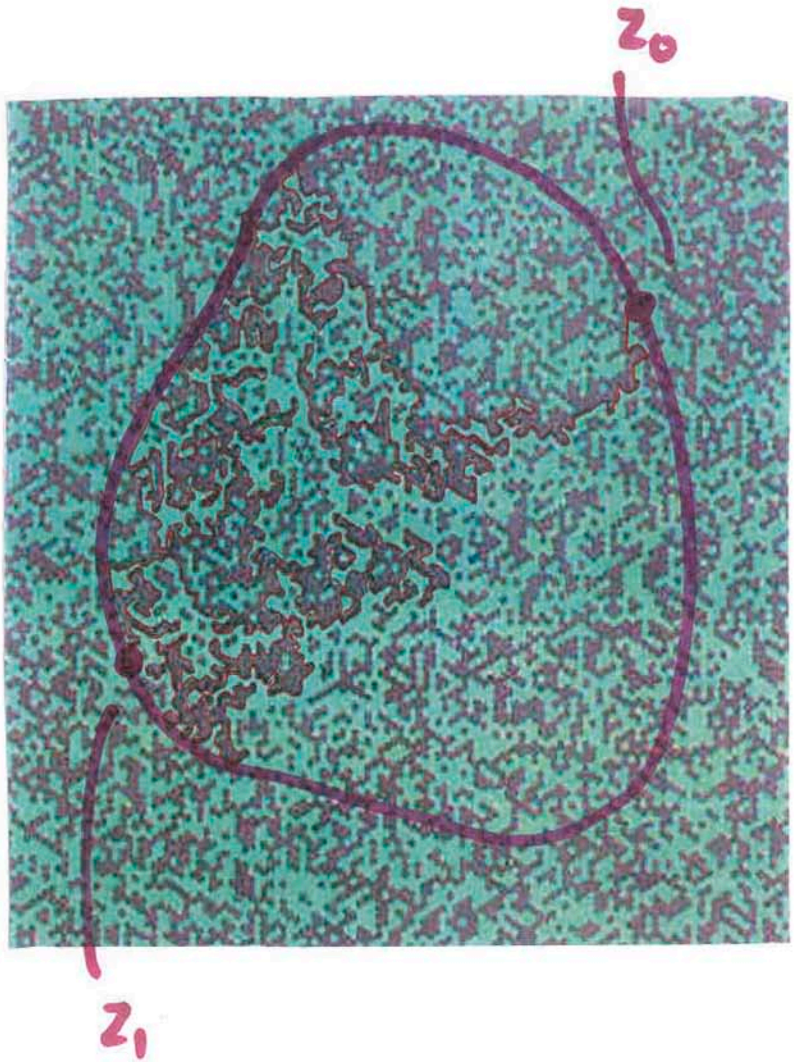
turning towards z_1 on hitting the boundary

We obtain a random chord.

The family of laws $(\mu_D^\varepsilon : D \in \mathcal{D})$ has the locality property.

Smirnov has shown that this family has a

conformally invariant limit as $\varepsilon \downarrow 0$.



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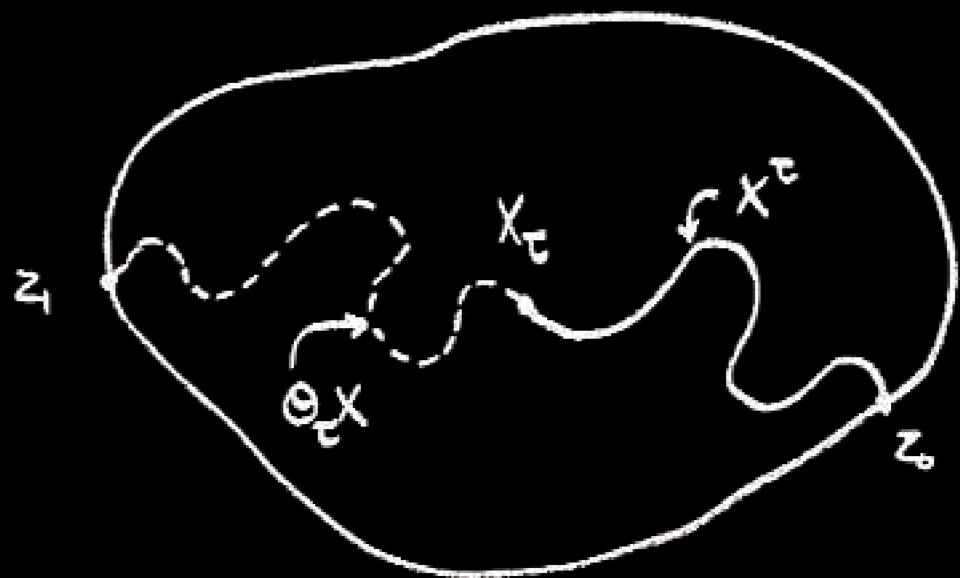
Domain Markov property



The idea is that the chord evolves randomly from τ to 1 with law μ_{D_τ} where

$$D_\tau = (U \setminus X^\tau, X_\tau, z_1).$$

Say $\tau : P_D \to [0,1]$ is parametrization-invariant if

$$\tau(\gamma \circ \phi) = \phi^{-1}(\tau(\gamma)), \quad \gamma \in P_D, \ \phi \text{ reparametrization}$$

For $x = [\gamma] \in C_D$, we then define

$$x_\tau = \gamma_{\tau(\gamma)}, \quad x^\tau = [\gamma|_{[0,\tau(\gamma)]}], \quad \Theta_\tau x = [\gamma|_{[\tau(\gamma),1]}]$$

stopped chord — shifted chord

Example $\tau(\gamma) = \inf\{t \geq 0 : \gamma_t \in B\}$

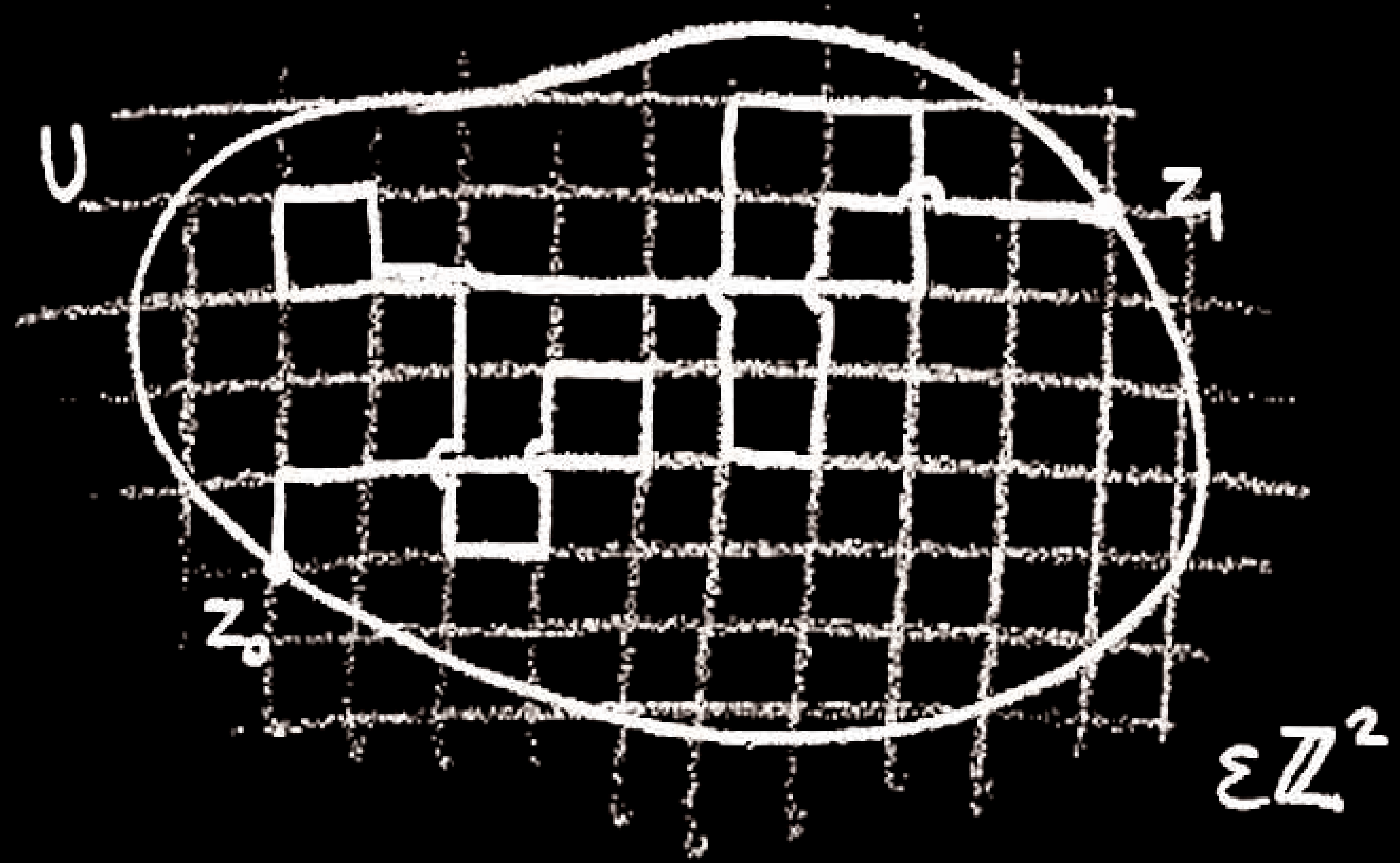
Provided x^τ does not hit z_1, there is a unique connected component $U_\tau = U_\tau(x)$ of $U \setminus \gamma[0,\tau(\gamma)]$ containing z_1.

The inclusion $U_\tau \to U$ extends to a continuous map $\hat{U}_\tau \to \hat{U}$.

We use this to think of μ_{D_τ} as a measure on $C_{(U, X_\tau, z_1)}$.

Say that $(\mu_D : D \in \mathcal{D})$ has the <u>domain Markov property</u> if, for all $D = (U, z_0, z_1) \in \mathcal{D}$, for $X \sim \mu_D$, X does not hit z_1 before time 1, almost surely, and for all parametrization-invariant stopping times τ on P_D, conditional on the stopped chord X^τ and on $\tau(X) < 1$, we have $\Theta_\tau X \sim \mu_{D_\tau}$.

Example: loop-erased random walk



Let $S_0, S_1, \ldots, S_T$ be a simple symmetric random walk on $\varepsilon\mathbb{Z}^2$ starting from z_0, run until it first leaves U.

Condition on $S_T = z_1$. Erase loops from S as follows:

$$T_0 = 0, \quad T_{n+1} = \sup\{m \le T : S_m = S_{T_n + 1}\}, \quad \tilde{S}_n = S_{T_n}.$$

Then $\tilde{S}_0, \tilde{S}_1, \ldots$ is a simple random chord in D.

The associated family of probability measures $(\mu_D^\varepsilon : D \in \mathcal{D})$ has (a suitable discrete version of) the domain Markov property.

This was a key clue in Schramm's discovery of SLE in 1999.

Lawler, Schramm & Werner have since shown that $\mu_D^\varepsilon \Rightarrow \mu_D$ as $\varepsilon \downarrow 0$, with $(\mu_D : D \in \mathcal{D})$ conformally invariant