Lectures on Schramm–Loewner Evolution

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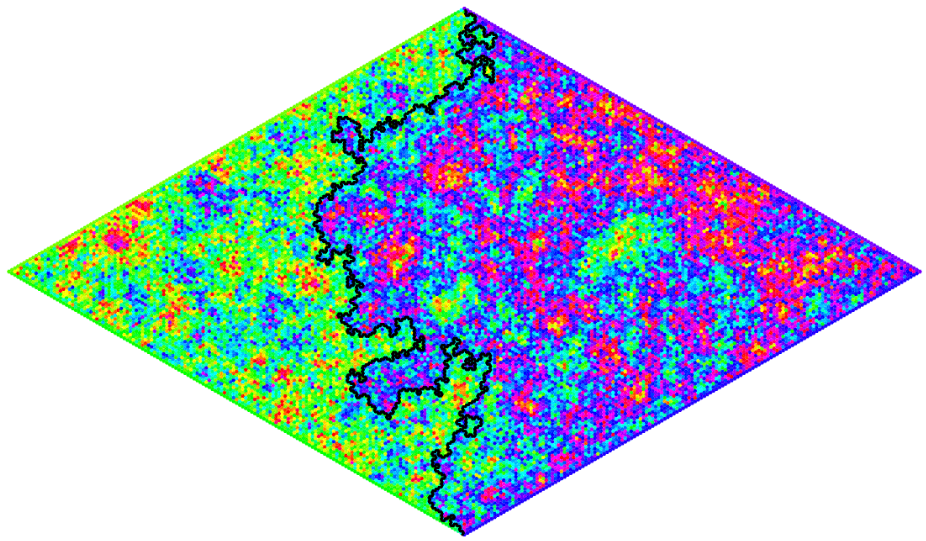
January 14, 2016



These notes are based on a course given to Masters students in Cambridge. Their scope is the basic theory of Schramm–Loewner evolution, together with some underlying and related theory for conformal maps and complex Brownian motion. The structure of the notes is influenced by our attempt to make the material accessible to students having a working knowledge of basic martingale theory and Itô calculus, whilst keeping the prerequisities from complex analysis to a minimum.

Contents

1 Riemann mapping theorem

We review the notion of conformal isomorphism of complex domains and discuss the question of existence and uniqueness of conformal isomorphisms between proper simply connected complex domains. Then we illustrate, by a simple special case, Loewner's idea of encoding the evolution of complex domains using a differential equation.

1.1 Conformal isomorphisms

We shall be concerned with certain sorts of subset of the complex plane $\mathbb{C}$ and mappings between them. A set $D \subseteq \mathbb{C}$ is a *domain* if it is non-empty, open and connected. We say that D is *simply connected* if every continuous map of the circle $\{|z| = 1\}$ into D is the restriction of a continuous map of the disc $\{|z| \leqslant 1\}$ into D. A convenient criterion for a domain $D \subseteq \mathbb{C}$ to be simply connected is that its complement in the Riemann sphere $\mathbb{C} \cup \{\infty\}$ is connected. A domain is *proper* if it is not the whole of $\mathbb{C}$. The open unit disc $\mathbb{D} = \{|z| < 1\}$, the open upper half-plane $\mathbb{H} = \{\mathrm{Re}(z) > 0\}$, and the open infinite strip $S = \{0 < \mathrm{Im}(z) < 1\}$ are all examples of proper simply connected domains.

A holomorphic function f on a domain D is a *conformal map* if $f'(z) \neq 0$ for all $z \in D$. We call a bijective conformal map $f : D \to D'$ a *conformal isomorphism*. In this case, the image $D' = f(D)$ is also a domain and the inverse map $f^{-1} : D' \to D$ is also a conformal map. Every conformal map is locally a conformal isomorphism. The function $z \mapsto e^z$ is conformal on $\mathbb{C}$ but is not a conformal isomorphism on $\mathbb{C}$ because it is not injective. We note the following fundamental result. A proof may be found in [1].

Theorem 1.1 (Riemann mapping theorem)**.** *Let D be a proper simply connected domain. Then there exists a conformal isomorphism $\phi : D \to \mathbb{D}$.*

We shall discuss ways to specify a unique choice of conformal isomorphism $\phi : D \to \mathbb{D}$ or $\phi : D \to \mathbb{H}$ in the next two sections. In general, there is no usable formula for ϕ in terms of D. Nevertheless, we shall want to derive certain properties of ϕ from properties of D. We shall see that Brownian motion provides a useful tool for this.

1.2 Möbius transformations

A *Möbius transformation* is any function f on $\mathbb{C} \cup \{\infty\}$ of the form

$$f(z) = \frac{az + b}{cz + d} \tag{1}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Here $f(-d/c) = \infty$ and $f(\infty) = a/c$. Möbius transformations form a group under composition. A Möbius transformation f restricts to a conformal automorphism of $\mathbb{H}$ if and only if we can write (1) with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. For $\theta \in [0, 2\pi)$ and $w \in \mathbb{D}$, define $\Phi_{\theta,w}$ on $\mathbb{D}$ by

$$\Phi_{\theta,w}(z) = e^{i\theta} \frac{z - w}{1 - \bar{w}z}.$$

Then $\Phi_{\theta,w}$ is a conformal automorphism of $\mathbb{D}$ and is the restriction of a Möbius transformation to $\mathbb{D}$. Define $\Psi : \mathbb{H} \to \mathbb{D}$ by

$$\Psi(z) = \frac{i-z}{i+z}.$$

Then Ψ is a conformal isomorphism and Ψ extends to a Möbius transformation. The following lemma is a basic result of complex analysis. We shall give a proof in Section 2.

Lemma 1.2 (Schwarz lemma)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $f(0) = 0$. Then $|f(z)| \leqslant |z|$ for all z. Moreover, if $|f(z)| = |z|$ for some $z \neq 0$, then $f(w) = e^{i\theta}w$ for all w, for some $\theta \in [0, 2\pi)$.*

Corollary 1.3. *Let ϕ be a conformal automorphism of $\mathbb{D}$. Set $w = \phi^{-1}(0)$ and $\theta = \arg \phi'(w)$. Then $\phi = \Phi_{\theta,w}$. In particular ϕ is the restriction of a Möbius transformation to $\mathbb{D}$ and extends to a homeomorphism of $\bar{\mathbb{D}}$.*

Proof. Set $f = \phi \circ \Phi_{0,w}^{-1}$. Then f is a conformal automorphism of $\mathbb{D}$ and $f(0) = 0$. Pick $u \in \mathbb{D} \setminus \{0\}$ and set $v = f(u)$. Note that $v \neq 0$. Now, either $|f(u)| = |v| \geqslant |u|$ or $|f^{-1}(v)| = |u| \geqslant |v|$. In any case, by the Schwarz lemma, there exists $\alpha \in [0, 2\pi)$ such that $f(z) = e^{i\alpha}z$ for all z, and so $\phi = f \circ \Phi_{0,w} = \Phi_{\alpha,w}$. Finally, $\Phi'_{\alpha,w}(w) = e^{i\alpha}/(1 - |w|^2)$ so $\alpha = \theta$. ☐

Corollary 1.4. *Let D be a proper simply connected domain and let $w \in D$. Then there exists a unique conformal isomorphism $\phi : D \to \mathbb{D}$ such that $\phi(w) = 0$ and $\arg \phi'(w) = 0$.*

Proof. By the Riemann mapping theorem there exists a conformal isomorphism $\phi_0 : D \to \mathbb{D}$. Set $v = \phi_0(w)$ and $\theta = -\arg \phi_0'(w)$ and take $\phi = \Phi_{\theta,v} \circ \phi_0$. Then $\phi : D \to \mathbb{D}$ is a conformal isomorphism with $\phi(w) = 0$ and $\arg \phi'(w) = 0$. If ψ is another such, then $f = \psi \circ \phi^{-1}$ is a conformal automorphism of $\mathbb{D}$ with $f(0) = 0$ and $\arg f'(0) = 0$, so $f = \Phi_{0,0}$ which is the identity function. Hence ϕ is unique. ☐

1.3 Martin boundary

The Martin boundary is a general object of potential theory[1]. We shall however limit our discussion to the case of harmonic functions in a proper simply connected complex domain D. In this case, the Riemann mapping theorem, combined with the conformal invariance of harmonic functions, allows a very simple approach. Make a choice of conformal isomorphism $\phi : D \to \mathbb{D}$. We can define a metric d_ϕ on D by $d_\phi(z, z') = |\phi(z) - \phi(z')|$. Then d_ϕ is locally equivalent to the original metric but possibly not uniformly so. Say that a sequence $(z_n : n \in \mathbb{N})$ in D is *D-Cauchy* if it is Cauchy for d_ϕ. Since every conformal automorphism of $\mathbb{D}$ extends to a homeomorphism of $\bar{\mathbb{D}}$, this notion does not depend on the choice of ϕ. Write $\hat{D}$ for the completion of D with respect to the metric[2] and define the *Martin boundary* $\delta D = \hat{D} \setminus D$. The set $\hat{D}$ does not depend on the choice of ϕ and

[1]See for example [2]

[2]This is the set of equivalence classes of D-Cauchy sequences $z = (z_n : n \in \mathbb{N})$, where $z \sim z'$ if $(z_1, z_1', z_2, z_2', \dots)$ is also a D-Cauchy sequence.

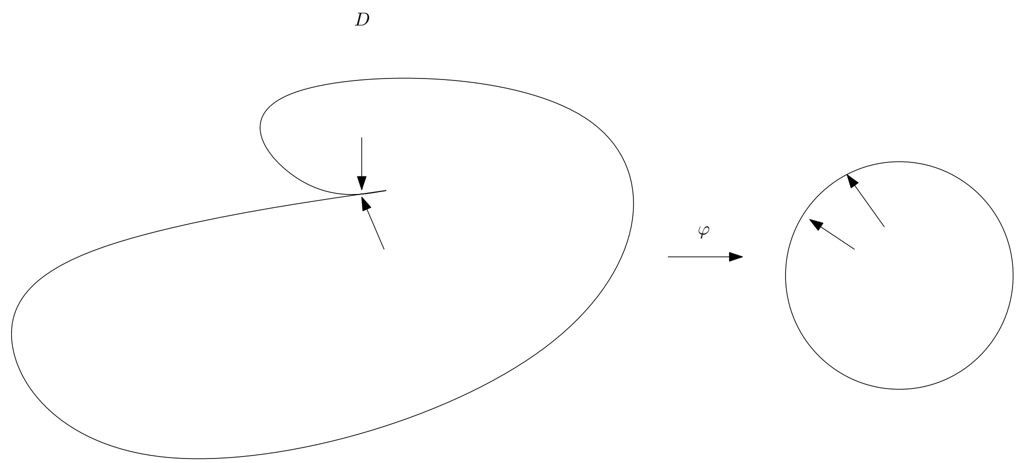


Figure 1: Two distinct points of δD and their images under φ.

nor does its topology. This construction ensures that the map ϕ extends uniquely to a homeomorphism $\hat{D} \to \bar{\mathbb{D}}$. It follows then that every conformal isomorphism ψ of proper simply connected domains $D \to D'$ has a unique extension as a homeomorphism $\hat{D} \to \hat{D}'$. We abuse notation in writing $\phi(z)$ for the value of this extension at points $z \in \delta D$. Write ∂D for the boundary of D as a subset of $\mathbb{C}$, that is the set of limit points of D in $\mathbb{C}$, which in general is not identifiable with δD. For $b \in \delta D$, we say that a simply connected subdomain $N \subseteq D$ is a *neighbourhood of b in D* if $\{z \in \mathbb{D} : |z - \phi(b)| < \varepsilon\} \subseteq \phi(N)$ for some $\varepsilon > 0$.

A *Jordan curve* is a continuous injective map $\gamma : \partial\mathbb{D} \to \mathbb{C}$. Say D is a *Jordan domain* if ∂D is the image of a Jordan curve. It can be shown in this case that any conformal isomorphism $D \to \mathbb{D}$ extends to a homeomorphism $\bar{D} \to \bar{\mathbb{D}}$, so we can identify δD with ∂D. On the other hand, a sequence $(z_n : n \in \mathbb{N})$ in $\mathbb{H}$ is $\mathbb{H}$-Cauchy if either it converges in $\mathbb{C}$ or $|z_n| \to \infty$ as $n \to \infty$. Thus we identify $\delta\mathbb{H}$ with $\mathbb{R} \cup \{\infty\}$. For the slit domain $D = \mathbb{H} \setminus (0, i]$ and, for $z \in [0, i)$, the sequences $(z + (1 + i)/n : n \in \mathbb{N})$ and $(z + (-1 + i)/n : n \in \mathbb{N})$ are D-Cauchy but are not equivalent, so their equivalence classes z^+ and z^- are distinct Martin boundary points.

Corollary 1.5. *Let ϕ be a conformal automorphism of $\mathbb{H}$. If $\phi(\infty) = \infty$, then $\phi(z) = \sigma z + \mu$ for all $z \in \mathbb{H}$, for some $\sigma > 0$ and $\mu \in \mathbb{R}$. If $\phi(\infty) = \infty$ and $\phi(0) = 0$, then $\phi(z) = \sigma z$ for all $z \in \mathbb{H}$, for some $\sigma > 0$.*

Proof. Set $\mu = \phi(0)$ and $\sigma = \phi(1) - \phi(0)$. Since $\Psi \circ \phi \circ \Psi^{-1}$ is a conformal automorphism of $\mathbb{D}$, we know by Corollary 1.3 that ϕ is a Möbius transformation of $\mathbb{H}$, so $\phi(z) = (az + b)/(cz + d)$ for all $z \in \mathbb{H}$, for some $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$. This formula extends by continuity to $\delta\mathbb{H} = \mathbb{R} \cup \{\infty\}$. So we must have $c = 0$, $\mu = b/d$ and $\sigma = a/d > 0$. □

Corollary 1.6. *Let D be a proper simply connected domain and let $b_1, b_2, b_3 \in \delta D$, ordered anticlockwise. Then there exists a unique conformal isomorphism $\phi : D \to \mathbb{H}$ such that $\phi(b_1) = 0$, $\phi(b_2) = 1$ and $\phi(b_3) = \infty$.*

Proof. By the Riemann mapping theorem there exists a conformal isomorphism $\phi_0 : D \to \mathbb{D}$. Set $\theta = \pi - \arg \phi_0(b_3)$ and take $\phi_1 = \Psi^{-1} \circ \Phi_{\theta,0} \circ \phi_0$. Then $\phi_1 : D \to \mathbb{H}$ is a conformal isomorphism, and $\Phi_{\theta,0} \circ \phi_0(b_3) = -1$ so $\phi_1(b_3) = \infty$. Now $\phi_1(b_1) < \phi_1(b_2)$ so there exist $\sigma \in (0, \infty)$ and $\mu \in \mathbb{R}$ such that $\sigma\phi_1(b_1)+\mu = 0$ and $\sigma\phi_1(b_2)+\mu = 1$. Set $\phi(z) = \sigma\phi_1(z)+\mu$ then $\phi : D \to \mathbb{H}$ is a conformal isomorphism satisfying the given constraints. If ψ is another such then $f = \psi \circ \phi^{-1}$ is a conformal automorphism of $\mathbb{H}$ with $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$. Hence $f(z) = z$ for all $z \in \mathbb{H}$ and so ϕ is unique. □

Note that in both Corollary 1.4 and Corollary 1.6, we obtain uniqueness of the conformal map by the imposition of three real-valued constraints.

1.4 SLE(0)

This section and the next are for orientation and do not form part of the theoretical development. Consider the (deterministic) process $(\gamma_t)_{t\geqslant 0}$ in the closed upper half-plane $\bar{\mathbb{H}}$ given by

$$\gamma_t = 2i\sqrt{t}.$$

This process belongs to the family of processes $(\mathrm{SLE}(\kappa) : \kappa \in [0, \infty))$ to which these notes are devoted, corresponding to the parameter value $\kappa = 0$. Think of $(\gamma_t)_{t\geqslant 0}$ as progressively eating away the upper half-plane so that the subdomain $H_t = \mathbb{H} \setminus K_t$ remains at time t, where $K_t = \gamma(0, t] = \{\gamma_s : s \in (0, t]\}$. There is a conformal isomorphism $g_t : H_t \to \mathbb{H}$ given by

$$g_t(z) = \sqrt{z^2 + 4t}$$

which has the following asymptotic behaviour as $|z| \to \infty$

$$g_t(z) = z + \frac{2t}{z} + O(|z|^{-2}).$$

In particular $g_t(z) - z \to 0$ as $|z| \to \infty$. As we shall show in Proposition 4.3, there is only one conformal isomorphism $H_t \to \mathbb{H}$ with this last property. Thus we can think of the family of maps $(g_t)_{t\geqslant 0}$ as a canonical encoding of the path $(\gamma_t)_{t\geqslant 0}$.

Consider the vector field b on $\bar{\mathbb{H}} \setminus \{0\}$ defined by

$$b(z) = \frac{2}{z} = \frac{2(x - iy)}{x^2 + y^2}.$$

Fix $z \in \bar{\mathbb{H}} \setminus \{0\}$ and define

$$\zeta(z) = \inf\{t \geqslant 0 : \gamma_t = z\} = \begin{cases} y^2/4, & \text{if } z = iy \\ \infty, & \text{otherwise.} \end{cases}$$

Then $\zeta(z) > 0$, and $z \in \bar{K}_t$ if and only if $\zeta(z) \leqslant t$. Set $z_t = g_t(z)$. Then for $t < \zeta(z)$

$$\dot{z}_t = \frac{2}{\sqrt{z_t^2 + 4t}} = b(z_t) \tag{2}$$

and, if $\zeta(z) < \infty$, then $z_t \to 0$ as $t \to \zeta(z)$. Thus $(g_t(z) : z \in \bar{\mathbb{H}} \setminus \{0\}, t < \zeta(z))$ is the (unique) maximal flow of the vector field b in $\bar{\mathbb{H}} \setminus \{0\}$. By *maximal* we mean that $(z_t : t < \zeta(z))$ cannot be extended to a solution of the differential equation on a longer time interval.

1.5 Loewner evolutions

Think of SLE(0) as obtained via the associated flow $(g_t)_{t \geqslant 0}$ by iterating continuously a map $g_{\delta t}$, which nibbles an infinitesimal piece $(0, 2i\sqrt{\delta t}]$ of $\mathbb{H}$ near 0. Charles Loewner, in the 1920's, studied complex domains $H_t = \mathbb{H} \setminus \gamma(0, t]$ for more general curves $(\gamma_t)_{t \geqslant 0}$, by a similar continuous iteration of conformal maps, obtained now by considering the flow of a time-dependent vector field $\bar{\mathbb{H}}$ of the form

$$b(t, z) = \frac{2}{z - \xi_t}, \quad t \geqslant 0, \quad z \in \mathbb{H}.$$

Here, $(\xi_t : t \geqslant 0)$ is a given continuous real-valued function, which is called the *driving function* or *Loewner transform* of the curve γ. We shall study this flow in detail below, showing that it always provides a construction of a family of domains $(H_t : t \geqslant 0)$, and sometimes also a path γ. Note that the flow lines $(g_t(z))_{t \geqslant 0}$ for SLE(0) separate, left and right, each side of the singularity at 0, with the path $(\gamma_t)_{t \geqslant 0}$ growing up between the left-moving flow lines and the right-moving ones. In the general case, assuming that the qualitative picture remains the same, when we move the singularity point ξ_t to the left, we may expect that some left-moving flow lines are deflected to the right, so the curve $(\gamma_t)_{t \geqslant 0}$ turns to the left. Moreover, the wilder the fluctuations of $(\xi_t)_{t \geqslant 0}$, the more convoluted we may expect the resulting path $(\gamma_t)_{t \geqslant 0}$ to be.

Oded Schramm, in 1999, realized that for some conjectured conformally invariant scaling limits $(\gamma_t)_{t \geqslant 0}$ of planar random processes, with a certain spatial Markov property, the process $(\xi_t)_{t \geqslant 0}$ would have to be a Brownian motion, of some diffusivity κ. The associated processes $(\gamma_t)_{t \geqslant 0}$ were at that time totally new and have since revolutionized our understanding of conformally invariant planar random processes.

2 Brownian motion and harmonic functions

We first prove a conformal invariance property of complex Brownian motion, due to Lévy. Then we prove Kakutani's formula relating Brownian motion and harmonic functions, and deduce from this the maximum principle for harmonic functions and the maximum modulus principle for holomorphic functions. This used to prove the Schwarz lemma.

2.1 Conformal invariance of Brownian motion

Theorem 2.1. *Let D and D' be domains and let $\phi : D \to D'$ be a conformal isomorphism. Fix $z \in D$ and set $z' = \phi(z)$. Let $(B_t)_{t\geqslant 0}$ and $(B'_t)_{t\geqslant 0}$ be complex Brownian motions starting from z and z' respectively. Set*

$$T = \inf\{t \geqslant 0 : B_t \notin D\}, \quad T' = \inf\{t \geqslant 0 : B'_t \notin D'\}.$$

Set $\tilde{T} = \int_0^T |\phi'(B_t)|^2 dt$ and define for $t < \tilde{T}$

$$\tau(t) = \inf\left\{s \geqslant 0 : \int_0^s |\phi'(B_r)|^2 dr = t\right\}, \quad \tilde{B}_t = \phi(B_{\tau(t)}).$$

Then $(\tilde{T}, (\tilde{B}_t)_{t<\tilde{T}})$ and $(T', (B'_t)_{t<T'})$ have the same distribution.

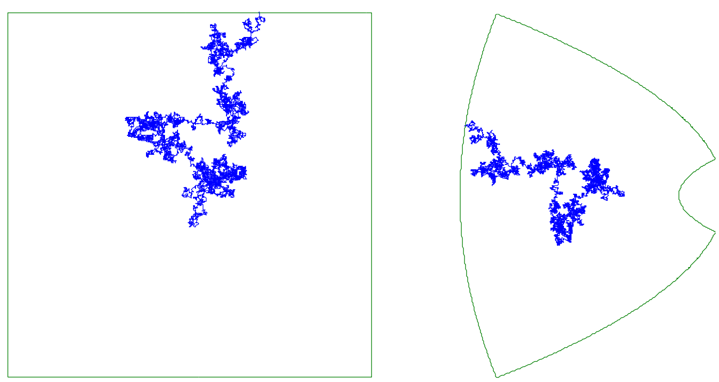


Figure 2: A Brownian motion stopped upon leaving the unit square, and its image under a conformal transformation

Proof. Assume for now that D is bounded and ϕ has a C^1 extension to $\bar{D}$. Then $T < \infty$ almost surely and we may define a continuous semimartingale[3] Z and a continuous adapted

[3]Here and below, where we use notions depending on a choice of filtration, such as martingale or stopping time, unless otherwise stated, these are to be understood with respect to the natural filtration $(\mathcal{F}_t)_{t\geqslant 0}$ of $(B_t)_{t\geqslant 0}$.

process A by setting[4]

$$Z_t = \phi(B_{T\wedge t}) + (B_t - B_{T\wedge t}), \quad A_t = \int_0^{T\wedge t} |\phi'(B_s)|^2 ds + (t - (T \wedge t)).$$

Moreover, almost surely, A is an (increasing) homeomorphism of $[0,\infty)$, whose inverse is an extension of τ. Denote the inverse homeomorphism also by τ. Write $\phi = u + iv$, $B_t = X_t + iY_t$ and $Z_t = M_t + iN_t$. By Itô's formula, for $t < T$,

$$dM_t = \frac{\partial u}{\partial x}(B_t)dX_t + \frac{\partial u}{\partial y}(B_t)dY_t, \quad dN_t = \frac{\partial v}{\partial x}(B_t)dX_t + \frac{\partial v}{\partial y}(B_t)dY_t$$

and so, using the Cauchy–Riemann equations,

$$dM_t dM_t = |\phi'(B_t)|^2 dt = dA_t = dN_t dN_t, \quad dM_t dN_t = 0.$$

On the other hand, for $t \geqslant T$,

$$dM_t = dX_t, \quad dN_t = dY_t, \quad dM_t dM_t = dt = dA_t = dN_t dN_t, \quad dM_t dN_t = 0.$$

Hence $(M_t)_{t\geqslant 0}$, $(N_t)_{t\geqslant 0}$, $(M_t^2 - A_t)_{t\geqslant 0}$, $(N_t^2 - A_t)_{t\geqslant 0}$ and $(M_t N_t)_{t\geqslant 0}$ are all continuous local martingales. Set $\tilde{M}_s = M_{\tau(s)}$ and $\tilde{N}_s = N_{\tau(s)}$. Then, by optional stopping, $(\tilde{M}_s)_{s\geqslant 0}$, $(\tilde{N}_s)_{s\geqslant 0}$, $(\tilde{M}_s^2 - s)_{s\geqslant 0}$, $(\tilde{N}_s^2 - s)_{s\geqslant 0}$ and $(\tilde{M}_s\tilde{N}_s)_{s\geqslant 0}$ are continuous local martingales for the filtration $(\tilde{\mathcal{F}}_s)_{s\geqslant 0}$, where $\tilde{\mathcal{F}}_s = \mathcal{F}_{\tau(s)}$. Define $(\tilde{Z}_s)_{s\geqslant 0}$ by $\tilde{Z}_s = \tilde{M}_s + i\tilde{N}_s$. Then, by Lévy's characterization of Brownian motion, $(\tilde{Z}_s)_{s\geqslant 0}$ is a complex $(\tilde{\mathcal{F}}_s)_{s\geqslant 0}$-Brownian motion starting from $z' = \phi(z)$. Now $\tilde{B}_t = \tilde{Z}_t$ for $t < \tilde{T}$ and, since ϕ is a bijection, $\tilde{T} = \inf\{t \geqslant 0 : \tilde{Z}_t \notin D'\}$. So we have shown the claimed identity of distributions.

In the cases where D is not bounded or ϕ fails to have a C^1 extension to $\bar{D}$, choose a sequence of bounded open sets $D_n \uparrow D$ with $\bar{D}_n \subseteq D$ for all n. Set $D'_n = \phi(D_n)$ and set

$$T_n = \inf\{t \geqslant 0 : B_t \notin D_n\}, \quad T'_n = \inf\{t \geqslant 0 : B'_t \notin D'_n\}.$$

Set $\tilde{T}_n = \int_0^{T_n} |\phi'(B_t)|^2 dt$. Then $\tilde{T}_n \uparrow \tilde{T}$ and $T'_n \uparrow T'$ almost surely as $n \to \infty$. Since ϕ is C^1 on $\bar{D}_n$, we know that $(\tilde{T}_n, (\tilde{B}_t)_{t<\tilde{T}_n})$ and $(T'_n, (B'_t)_{t<T'_n})$ have the same distribution for all n, which implies the desired result on letting $n \to \infty$. □

Corollary 2.2. *Let D be a proper simply connected domain. Fix $z \in D$ and let $(B_t)_{t\geqslant 0}$ be a complex Brownian motion starting from z. Set $T(D) = \inf\{t \geqslant 0 : B_t \notin D\}$. Then $\mathbb{P}_z(T(D) < \infty) = 1$.*

Proof. There exists a conformal isomorphism $\phi : D \to \mathbb{D}$. By conformal invariance of Brownian motion $(\phi(B_t) : t < T(D))$ is a time-change of Brownian motion run up to the finite time when it first exits from $\mathbb{D}$. Hence $|\phi(B_t)| \geqslant 1/2$ eventually as $t \uparrow T(D)$. But $(B_t)_{t\geqslant 0}$ is neighbourhood recurrent so visits the open set $\{z \in D : |\phi(z)| < 1/2\}$ at an unbounded set of times almost surely. Hence $T(D) < \infty$ almost surely. □

[4]Whereas Itô calculus localizes nicely with respect to stopping times, and we exploit this, Lévy's characterization of Brownian motion is usually formulated globally. The extension of Z and A beyond the exit time T exploits the robustness of Itô calculus to set up for an application of Lévy's characterization without localization.

2.2 Kakutani's formula and the circle average property

A real-valued function u defined on a domain $D \subseteq \mathbb{C}$ is *harmonic* if u is twice continuously differentiable on D with

$$\Delta u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u = 0$$

everywhere on D. Harmonic functions can often be recovered from their boundary values using Brownian motion.

Theorem 2.3 (Kakutani's formula)**.** *Let u be a harmonic function defined on a bounded domain D and having a continuous extension to the closure $\bar{D}$. Fix $z \in D$ and let $(B_t)_{t\geqslant 0}$ be a complex Brownian motion starting from z. Set $T(D) = \inf\{t \geqslant 0 : B_t \notin D\}$. Then*

$$u(z) = \mathbb{E}_z(u(B_{T(D)})).$$

Proof. Suppose for now that u is the restriction to D of a C^2 function on $\mathbb{C}$. Denote this function also by u. Define $(M_t)_{t\geqslant 0}$ by the Itô integral

$$M_t = u(z) + \int_0^t \nabla u(B_s) dB_s.$$

Then $(M_t)_{t\geqslant 0}$ is a continuous local martingale. By Itô's formula, $u(B_t) = M_t$ for all $t \leqslant T$. Hence the stopped process M^T is uniformly bounded and, by optional stopping,

$$u(z) = M_0 = \mathbb{E}_z(M_T) = \mathbb{E}_z(u(B_{T(D)})).$$

For each $n \in \mathbb{N}$, the restriction of u to $D_n = \{z \in D : \mathrm{dist}(z, \partial D) > 1/n\}$ has a C^2 extension to $\mathbb{C}$, regardless of whether u itself does. The preceding argument then shows that $u(z) = \mathbb{E}_z(u(B_{T(D_n)}))$ for all $z \in D_n$. Now $T(D_n) \uparrow T(D) < \infty$ as $n \to \infty$ almost surely. Since B is continuous and u extends continuously to $\bar{D}$, we obtain the desired identity by bounded convergence on letting $n \to \infty$. □

In fact, the validity of Kakutani's formula, even just in the special case where D is a disc centred at z, turns out to be a useful characterization of harmonic functions. We will use the following result in Section 3. A proof may be found in [2].

Proposition 2.4. *Let D be a domain and let $u : D \to [0, \infty]$ be a measurable function. Suppose that u has the following* circle average property*: for all $z \in D$ and any $r \in (0, d(z, \partial D))$, we have*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta.$$

Then, either $u(z) = \infty$ for all $z \in D$, or u is harmonic.

2.3 Maximum principle

Kakutani's formula implies immediately that a harmonic function u on a bounded domain D, which extends continuously to $\bar{D}$, cannot exceed the supremum of its values on the boundary ∂D. Moreover, as we now show, a harmonic function cannot achieve a maximum value on its domain, unless it is constant.

Theorem 2.5 (Maximum principle)**.** *Let u be a harmonic function defined on a domain D. Suppose there exists a point $z \in D$ such that $u(w) \leqslant u(z)$ for all $w \in D$. Then u is constant.*

Proof. It will suffice to consider the case where u has a finite supremum value m, say, on D. Consider the set $D_0 = \{z \in D : u(z) = m\}$. Then D_0 is relatively closed in D, since u is continuous. On the other hand, if $z \in D_0$, then for $\varepsilon > 0$ sufficiently small, the disc $B(z, \varepsilon)$ of radius ε and centre z is contained in D. So, by Kakutani's formula

$$m = u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + \varepsilon e^{i\theta}) d\theta.$$

Since u is continuous and bounded above by m, this implies that $w \in D_0$ whenever $|w-v| = \varepsilon$. Hence D_0 is open. Since D is connected, D_0 can only be non-empty if it is the whole of D. □

By the Cauchy–Riemann equations, the real and imaginary parts of a holomorphic function are harmonic. Hence, if f is holomorphic on a bounded domain D and extends continuously to $\bar{D}$, then f may be recovered from its boundary values, just as in Kakutani's formula: for all $z \in D$

$$f(z) = \mathbb{E}_z(f(B_{T(D)}))$$

and we have the estimate

$$|f(z)| \leqslant \sup_{w \in \partial D} |f(w)|.$$

Then a small variation on the argument for the maximum principle leads to the following result.

Theorem 2.6 (Maximum modulus principle)**.** *Let f be a holomorphic function defined on a domain D. Suppose there exists a point $z \in D$ such that $|f(w)| \leqslant |f(z)|$ for all $w \in D$. Then f is constant.*

We can now prove Lemma 1.2.

Proof of the Schwarz lemma. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $f(0) = 0$. Consider the function $g(z) = f(z)/z$. By Taylor's theorem, g is analytic and hence holomorphic in $\mathbb{D}$. Fix $z \in \mathbb{D}$ and $r \in (|z|, 1)$. Then

$$|g(z)| \leqslant \sup_{|w|=r} |g(w)| \leqslant \frac{1}{r}.$$

Letting $r \to 1$, we get $|g(z)| \leqslant 1$ and hence $|f(z)| \leqslant |z|$ for all $z \in \mathbb{D}$. If $|f(z)| = |z|$ for some $z \neq 0$, then $|g(z)| = 1$, say $g(z) = e^{i\theta}$. Then g is constant on $\mathbb{D}$ by the maximum modulus principle, so $f(w) = e^{i\theta} w$ for all $w \in \mathbb{D}$. $\square$

3 Harmonic measure and the Green function

3.1 Harmonic measure

Harmonic measures are objects of potential theory. Here, we will consider harmonic measures in the particular context of planar domains, introducing them through their interpretation as the hitting distributions of Brownian motion. We further confine our attention to the case of proper simply connected domains. Let D be such a domain and let δD be its Martin boundary. Let $(B_t)_{t\geqslant 0}$ be a complex Brownian motion starting from $z \in D$ and consider the first exit time $T = T(D)$ as in Section 2.2. We have shown that $T < \infty$ almost surely. In the case $D = \mathbb{D}$ and $z = 0$, we know that B_t converges in $\bar{\mathbb{D}}$ as $t \uparrow T$, with limit B_T uniformly distributed on the unit circle. In general, there exists a conformal isomorphism $\phi : D \to \mathbb{D}$ taking z to 0. Then, by conformal invariance of Brownian motion, as $t \uparrow T$, B_t converges in $\hat{D}$ to a limit $\hat{B}_T \in \delta D$. Denote by $h_D(z, .)$ the distribution of $\hat{B}_T$ on δD. We call $h_D(z, .)$ the *harmonic measure for D starting from z*. By the argument used for Kakutani's formula, if u is a harmonic function on D which extends continuously to $\hat{D}$, then we can recover u from its boundary values by[5]

$$u(z) = \mathbb{E}_z(u(\hat{B}_T)) = \int_{\delta D} u(s) h_D(z, ds).$$

We can compute $h_D(z, .)$ as follows. By conformal invariance of Brownian motion, for $s_1, s_2 \in \delta D$ and $\theta_1, \theta_2 \in [0, 2\pi)$ with $\theta_1 \leqslant \theta_2$ and $\phi(s_1) = e^{i\theta_1}$ and $\phi(s_2) = e^{i\theta_2}$, we have

$$h_D(z, [s_1, s_2]) = \mathbb{P}_z(\hat{B}_T \in [s_1, s_2]) = \mathbb{P}_0(B_{T(\mathbb{D})} \in [e^{i\theta_1}, e^{i\theta_2}]) = \frac{\theta_2 - \theta_1}{2\pi}.$$

We often fix an interval $I \subseteq \mathbb{R}$ and a parametrization $s : I \to \delta D$ of the Martin boundary. We may then be able to find a density function $h_D(z, .)$ on I such that

$$\int_{t_1}^{t_2} h_D(z, t)\, dt = h_D(z, [s(t_1), s(t_2)])$$

If we determine θ as a continuous function on I such that $e^{i\theta(t)} = \phi(s(t))$, then[6]

$$h_D(z, t) = \frac{1}{2\pi} \frac{d\theta}{dt}.$$

The following two examples are not only for illustration but will also be used later.

Example 3.1. *Take $D = \mathbb{D}$ and parametrize the boundary as $(e^{it} : t \in [0, 2\pi))$. Fix $w = x + iy \in \mathbb{D}$ and recall from Section 1.3 the conformal automorphism $\Phi_{0,w}$ on $\mathbb{D}$ taking*

[5] This is not Kakutani's formula, unless D is a Jordan domain. For example, if $D = \mathbb{H} \setminus (0, i]$, then the requirement that u extend continuously to $\hat{D}$ imposes that u have a limit at ∞ but allows different boundary values on each side of the slit $[0, i)$.

[6] Note that the function θ on I is determined uniquely by D and s up to an additive constant.

w to 0. The boundary parametrizations are then related by $e^{i\theta} = (e^{it} - w)/(1 - \bar{w}e^{it})$. On differentiating with respect to t, we find an expression for $d\theta/dt$, and hence obtain

$$h_{\mathbb{D}}(w,t) = \frac{1}{2\pi}\frac{1-|w|^2}{|e^{it}-w|^2} = \frac{1}{2\pi}\frac{1-x^2-y^2}{(\cos t - x)^2 + (\sin t - y)^2}, \quad 0 \leqslant t < 2\pi.$$

Example 3.2. *Take $D = \mathbb{H}$ with the obvious parametrization of the boundary by $\mathbb{R}$. Fix $w = x + iy \in \mathbb{H}$ and consider the conformal isomorphism $\phi : \mathbb{H} \to \mathbb{D}$ taking w to 0 given by $\phi(z) = (z-w)/(z-\bar{w})$. The boundary parametrizations are related by $e^{i\theta} = (t-w)/(t-\bar{w})$, so*

$$h_{\mathbb{H}}(w,t) = \frac{1}{\pi}\operatorname{Im}\left(\frac{1}{t-w}\right) = \frac{y}{\pi((t-x)^2+y^2)}, \quad t \in \mathbb{R}.$$

3.2 An estimate for harmonic functions ($\star$)

We will mark with ($\star$) some sections and proofs which might be omitted on a first reading. The following lemma allows us to bound the partial derivatives of a harmonic function in terms of its supremum norm. In conjunction with the Cauchy–Riemann equations, this will later allow us to deduce estimates on a holomorphic function starting from estimates on its real part. We present it here to illustrate how explicit calculations of harmonic measure can be used as a tool to obtain general estimates.

Lemma 3.3. *Let u be a harmonic function in D and let $z \in D$. Then*

$$\left|\frac{\partial u}{\partial x}(z)\right| \leqslant \frac{4\|u\|_\infty}{\pi \operatorname{dist}(z, \partial D)}.$$

Proof. It will suffice to show that, for all $\varepsilon > 0$, the estimate holds with 4 replaced by $4(1+\varepsilon)$. Fix $\varepsilon > 0$. By scaling and translation, we reduce to the case where $z = 0$ and $\operatorname{dist}(0, \partial D) = 1 + \varepsilon$. Then u is continuous on $\bar{\mathbb{D}}$ so, for $z \in \mathbb{D}$,

$$u(z) = \int_0^{2\pi} u(e^{i\theta}) h_{\mathbb{D}}(z,\theta) d\theta.$$

On differentiating the formula for the harmonic density obtained in Example 3.1, we see that $\nabla h_{\mathbb{D}}(\cdot, \theta)$ is bounded on a neighbourhood of 0, uniformly in θ, with $\nabla h_{\mathbb{D}}(0,\theta) = (\cos\theta, \sin\theta)/\pi$. Hence we may differentiate under the integral sign to obtain

$$\nabla u(0) = \frac{1}{\pi}\int_0^{2\pi} u(e^{i\theta})(\cos\theta, \sin\theta) d\theta.$$

Then

$$\left|\frac{\partial u}{\partial x}(0)\right| \leqslant \frac{\|u\|_\infty}{\pi}\int_0^{2\pi} |\cos\theta| d\theta = \frac{4\|u\|_\infty}{\pi} = \frac{4(1+\varepsilon)\|u\|_\infty}{\pi \operatorname{dist}(0, \partial D)}.$$

□

3.3 Dirichlet heat kernel and the Green function

We give a probabilistic definition of these two functions associated to a domain D and derive some of their properties. This will be used later in our discussion of the Gaussian free field. It will be convenient to have the following regularity property for exit probabilities of the Brownian bridge.

Proposition 3.4. *Define for $t \in (0,\infty)$ and $x, y \in D$*

$$\pi_D(t,x,y) = \mathbb{P}\left(X_s \in D \textit{ for all } s \in [0,t]\right)$$

where $(X_s)_{0\leqslant s\leqslant t}$ be a Brownian bridge from x to y in time t. Then π_D symmetric in its second and third arguments and is jointly continuous in all three.

Proof. ($\star$) A simple scaling and translation allows us to vary the time and endpoints of the Brownian bridge. Thus, from a single Brownian bridge $(W_s)_{0\leqslant s\leqslant 1}$ in $\mathbb{R}^2$ from 0 to 0 in time 1, we can realise $(X_s)_{0\leqslant s\leqslant t}$ with explicit dependence on t, x and y by

$$X_s = (1-(s/t))x + (s/t)y + \sqrt{t}W_{s/t}, \quad 0 \leqslant s \leqslant t.$$

This makes clear that $\pi_D(t,x,y) = \pi_D(t,y,x)$, since $(W_s)_{0\leqslant s\leqslant 1}$ is time-reversible.

Fix $\varepsilon \in (0,1/2)$ and define

$$D(\varepsilon) = \{z \in D : \operatorname{dist}(z,\partial D) > \varepsilon\}, \quad \pi_D^{(\varepsilon)}(t,x,y) = \mathbb{P}\left(X_s \in D \text{ for all } s \in [\varepsilon t,(1-\varepsilon)t]\right).$$

Note that

$$\pi_D^{(\varepsilon)}(t,x,y) = \int_{D^2} \pi_D((1-2\varepsilon)t,x',y')\rho^{(\varepsilon)}(x',y')dx'dy'$$

where $\rho^{(\varepsilon)}$ is the joint density of $(X_{\varepsilon t}, X_{(1-\varepsilon)t})$. Then, for all sequences $t_n \to t$, $x_n \to x$, and $y_n \to y$, with obvious notation, we have

$$|X^n_{st_n} - X_{st}| \leqslant |x_n - x| + |y_n - y| + |\sqrt{t_n} - \sqrt{t}| \sup_{0\leqslant s\leqslant 1} |W_s|$$

and $\rho_n^{(\varepsilon_n)} \to \rho^{(\varepsilon)}$ in $L^1(D^2)$, where $\varepsilon_n \in (0,1/2)$ is determined by $(1-2\varepsilon_n)t_n = (1-2\varepsilon)t$ for n sufficiently large. Hence

$$\liminf_n \pi_D(t_n,x_n,y_n) \geqslant \pi_{D(\varepsilon)}(t,x,y)$$

and

$$\limsup_n \pi_D(t_n,x_n,y_n) \leqslant \lim_n \pi_D^{(\varepsilon_n)}(t_n,x_n,y_n) = \pi_D^{(\varepsilon)}(t,x,y).$$

But $\pi_{D(\varepsilon)}(t,x,y) \to \pi_D(t,x,y)$ and $\pi_D^{(\varepsilon)}(t,x,y) \to \pi_D(t,x,y)$ as $\varepsilon \to 0$. Hence π_D is continuous as claimed. $\square$

Define the *Dirichlet heat kernel* p_D on $(0,\infty)\times D\times D$ and the *Green function* G_D on $D\times D$ by

$$p_D(t,x,y) = p(t,x,y)\pi_D(t,x,y), \quad G_D(x,y) = \int_0^\infty p_D(t,x,y)dt \tag{3}$$

where $p(t,x,y) = (2\pi t)^{-1}e^{-|x-y|^2/(2t)}$. Note that $\pi_D(t,x,x) \to 1$ as $t \to 0$, so $G_D(x,x) = \infty$ for all $x \in D$. The Green function is related directly to Brownian motion as an expected occupation density: thus, for all $x \in D$ and all non-negative measurable functions f on D, we have

$$\int_D G_D(x,y)f(y)dy = \mathbb{E}_x \int_0^{T(D)} f(B_t)dt. \tag{4}$$

This follows from the definition using Fubini's theorem and is left as an exercise.

Extend G_D by 0 outside $D\times D$. Then, for any sequence of domains $D_n \uparrow D$, we have $G_{D_n}(x,y) \uparrow G_D(x,y)$ for all $x,y \in D$. This follows from the definition by monotone convergence and is left as an exercise.

We say that a domain D is *Greenian* if $G_D(x,y) < \infty$ for some $x,y \in D$.

Proposition 3.5. *Every bounded domain is Greenian. Moreover, for any Greenian domain D, the Green function G_D is finite and continuous on $\{(x,y) \in D\times D : x \neq y\}$.*

Proof. ($\star$) For D bounded, there is a constant $\lambda > 0$ such that $\pi_D(1,x,y) \leqslant e^{-\lambda}$ for all $x,y \in D$. Then, by the Markov property, $\pi_D(t,x,y) \leqslant e^{-\lambda(t-1)}$ for all t, so $p_D(t,x,y) \leqslant e^{\lambda}t^{-1}e^{-\lambda t}e^{-\delta^2/(2t)}$ whenever $|x-y| \geqslant \delta$. Then, by Proposition 3.4 and dominated convergence, G_D is finite and continuous away from the diagonal. In particular D is Greenian.

Fix $y \in D$ and choose a sequence of probability density functions f_n, with f_n supported in $\{z \in D : |z-y| \leqslant 1/n\}$. Set $g_n(x) = \int_D G_D(x,z)f_n(z)dz$. Then g_n is a finite non-negative measurable function on D and $g_n \to G_D(.,y)$ as $n \to \infty$ locally uniformly on $D\setminus\{y\}$ by continuity. Moreover, using the identity (4) and the strong Markov property, g_n has the circle average property on $\{z \in D : |z-y| > 1/n\}$. Hence $G_D(.,y)$ has the circle average property on $D\setminus\{y\}$.

Now take any domain D, fix $y \in D$ and choose bounded domains $D_n \uparrow D$. Then $G_{D_n} \uparrow G_D$ so, by monotone convergence, $G_D(.,y)$ has the circle average property on $D\setminus\{y\}$. Hence $G_D(.,y)$ is either identically infinite or harmonic on $D\setminus\{y\}$. Then, if D is Greenian, we can use symmetry to see that G_D is finite and continuous on $\{(x,y) \in D\times D : x \neq y\}$. □

Conformal invariance of Brownian motion leads to a simple conformal invariance property for the Green function.

Proposition 3.6. *Let $\phi : D \to \phi(D)$ be a conformal isomorphism of planar domains. Then*

$$G_{\phi(D)}(\phi(x),\phi(y)) = G_D(x,y), \quad x,y \in D.$$

Proof. Since $G_{D_n} \uparrow G_D$ for $D_n \uparrow D$, it will suffice to consider the case where D and $\phi(D)$ are bounded when we know that G_D and $G_{\phi(D)}$ are finite and continuous away from the diagonal. Fix $x \in D$ and a non-negative measurable function f on D. Set $g = (f \circ \phi)|\phi'|^2$. Then, by the Jacobian formula,

$$\int_D G_{\phi(D)}(\phi(x), \phi(y))g(y)dy = \int_{\phi(D)} G_{\phi(D)}(\phi(x), w)f(w)dw$$
$$= \mathbb{E}_{\phi(x)} \int_0^{T(\phi(D))} f(B_t)dt = \mathbb{E}_x \int_0^{T(D)} g(B_\tau)d\tau = \int_D G_D(x,y)g(y)dy.$$

Hence $G_{\phi(D)}(\phi(x), \phi(y)) = G_D(x,y)$ for all $y \in D$. $\square$

For the upper half-plane, we can calculate explicity using the reflection principle

$$p_{\mathbb{H}}(t,x,y) = p(t,x,y) - p(t,\bar{x},y), \quad x,y \in \mathbb{H}.$$

Then the integral (3) can be evaluated, using the formula $e^{-a/t} - e^{-b/t} = t^{-1}\int_a^b e^{-x/t}dx$ and Fubini, to obtain

$$G_{\mathbb{H}}(x,y) = \frac{1}{\pi}\log\left|\frac{y-\bar{x}}{y-x}\right|.$$

Then, by conformal invariance, every proper simply connected domain is Greenian. Also, by a suitable choice of ϕ, we get the following simple formula for the Green function of the unit disc

$$G_{\mathbb{D}}(0,y) = -\frac{\log|y|}{\pi}, \quad y \in \mathbb{D}.$$

4 Compact $\mathbb{H}$-hulls and their mapping-out functions

A subset K of the upper half-plane $\mathbb{H}$ is called a *compact* $\mathbb{H}$*-hull* if K is bounded and $H = \mathbb{H} \setminus K$ is a simply connected domain. We shall associate to K a canonical conformal isomorphism $g_K : H \to \mathbb{H}$, the *mapping-out function* of K. At the same time we associate to K a real constant a_K, which we will identify later, in Section 6.2, as the *half-plane capacity* of K. These are all basic objects of Loewner's theory, or more precisely of its *chordal* variant, where we consider evolution of hulls in a given domain towards a chosen boundary point. We shall see later that the theory has a property of conformal invariance which allows us to reduce the general case to the study of the special domain $\mathbb{H}$ with ∞ as the boundary point, which is mathematically most tractable.

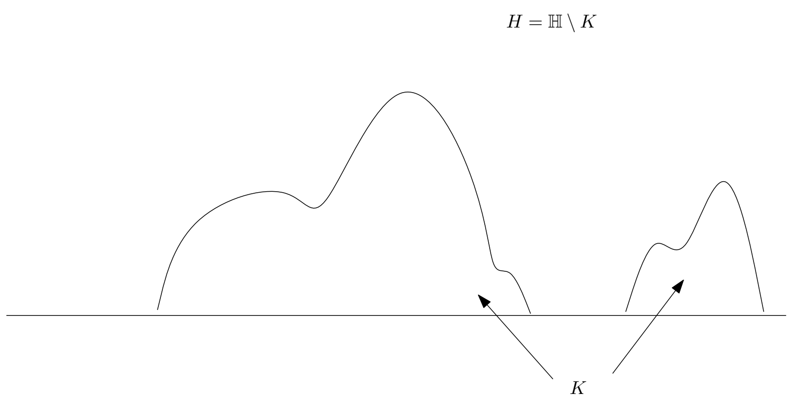


Figure 3: A compact $\mathbb{H}$-hull.

4.1 Extension of conformal maps by reflection

We start by explaining how a conformal isomorphism $\phi : D \to \mathbb{H}$ can be extended analytically to suitably regular parts of the boundary ∂D. We have already seen that ϕ extends continuously to the Martin boundary but now we want more regularity. The idea is to reflect the domain across the boundary. Given a proper simply connected domain $D \subseteq \mathbb{H}$, define

$$D^0 = \{x \in \mathbb{R} : D \text{ is a neighbourhood of } x \text{ in } \mathbb{H}\}, \quad D^* = D \cup D^0 \cup \{\bar{z} : z \in D\}.$$

More generally, for any open set $U \subseteq D^0$, define

$$D^*_U = D \cup U \cup \{\bar{z} : z \in D\}.$$

As U varies, the sets D^*_U are exactly the open sets which are invariant under conjugation and whose intersection with $\mathbb{H}$ is D. Say that a function $f^* : D^*_U \to \mathbb{C}$ is *reflection-invariant* if

$$f^*(\bar{z}) = \overline{f^*(z)}, \quad z \in D^*_U.$$

Given a continuous function f on D, there is at most one continuous, reflection-invariant function f^* on D_U^* extending f. Then f^* is the *continuous extension by reflection* of f. Such an extension f^* exists exactly when f has a continuous extension to $D \cup U$ which is real-valued on U. Any continuous extension by reflection of a holomorphic function is holomorphic, by an application of Morera's theorem. This is called the Schwarz reflection principle.

Proposition 4.1. *Let $D \subseteq \mathbb{H}$ be a simply connected domain. Let I be a proper open subinterval of $\mathbb{R}$ with $I \subseteq D^0$ and let $x \in I$. Then there exists a unique conformal isomorphism $\phi : D \to \mathbb{H}$ which extends to a homeomorphism $D \cup I \to \mathbb{H} \cup (-1, 1)$ taking x to 0. In particular I is naturally identified with an interval of the Martin boundary δD. Moreover ϕ extends further to a reflection-invariant conformal isomorphism $\phi^* : D_I^* \to \mathbb{H}^*_{(-1,1)}$.*

Proof ($\star$). Note that D_I^* and $\mathbb{H}^*_{(-1,1)}$ are proper simply connected domains. By the Riemann mapping theorem, there exists a unique conformal isomorphism $\phi^* : D_I^* \to \mathbb{H}^*_{(-1,1)}$ with $\phi^*(x) = 0$ and $\arg(\phi^*)'(x) = 0$. Define $\rho : D_I^* \to \mathbb{H}^*_{(-1,1)}$ by $\overline{\rho(z)} = \phi^*(\bar{z})$. Then ρ is a conformal isomorphism with $\rho(x) = 0$ and $\arg \rho'(x) = 0$. Hence $\rho = \phi^*$ and so ϕ^* is reflection-invariant. Then $\phi^*(I) \subseteq (-1, 1)$ and $(\phi^*)^{-1}(-1, 1) \subseteq I$, so $\phi^*(I) = (-1, 1)$. Now $\phi^*(D)$ is connected and does not meet $(-1, 1)$. Since $\arg(\phi^*)'(x) = 0$, by considering a neighbourhood of x, we must have $\phi^*(D) \subseteq \mathbb{H}$. The same argument shows that $(\phi^*)^{-1}(\mathbb{H}) \subseteq D$, so $\phi^*(D) = \mathbb{H}$. Hence ϕ^* restricts to a conformal isomorphism $\phi : D \to \mathbb{H}$ with the required properties.

On the other hand, any map ψ with these properties has a continuous extension ψ^* by reflection to D_I^*, which is a bijection to $\mathbb{H}^*_{(-1,1)}$ and is holomorphic by the Schwarz reflection principle. Moreover $\psi^*(x) = 0$, and $\arg(\psi^*)'(x) = 0$ since $\psi^*(I) = (-1, 1)$. Hence $\psi^* = \phi^*$ and so $\psi = \phi$. $\square$

Proposition 4.2. *Let $D \subseteq \mathbb{H}$ be a simply connected domain and let $\phi : D \to \mathbb{H}$ be a conformal isomorphism. Suppose that ϕ is bounded on bounded sets. Then ϕ extends by reflection to a conformal isomorphism ϕ^* on D^*.*

Proof ($\star$). Fix $x \in D^0$ and a bounded open interval $I \subseteq D^0$ containing x. Write $\phi_{x,I}$ for the conformal isomorphism obtained in Proposition 4.1. Then $f = \phi \circ \phi_{x,I}^{-1} : \mathbb{H} \to \mathbb{H}$ is a Möbius transformation which is bounded, and hence continuous, on a neighbourhood of $(-1, 1) = \phi_{x,I}(I)$ in $\mathbb{H}$. Hence $\phi = f \circ \phi_{x,I}$ extends by reflection to a conformal isomorphism $\phi_I^* = f^* \circ \phi_{x,I}^*$ on D_I^*. The maps ϕ_I^* must be consistent, and hence extend to a conformal map ϕ^* on D^*. Now ϕ^* can only fail to be injective on D^0 but, as a conformal map, can only fail to be injective on an open set in $\mathbb{C}$. Hence ϕ^* is a conformal isomorphism. $\square$

4.2 Construction of the mapping-out function

Given any compact $\mathbb{H}$-hull K, we now specify a particular conformal isomorphism $g = g_K : \mathbb{H} \setminus K \to \mathbb{H}$. This will give us a convenient way to encode the geometry of K. We get uniqueness by requiring that g_K looks like the identity at ∞.

Theorem 4.3. *Let K be a compact $\mathbb{H}$-hull and set $H = \mathbb{H} \setminus K$. There exists a unique conformal isomorphism $g_K : H \to \mathbb{H}$ such that $g_K(z) - z \to 0$ as $|z| \to \infty$. Moreover $g_K(z) - z$ is bounded uniformly in $z \in H$. Moreover, for some $a_K \in \mathbb{R}$, we have*

$$g_K(z) = z + \frac{a_K}{z} + O(|z|^{-2}), \quad |z| \to \infty. \tag{5}$$

Moreover g_K extends by reflection to a conformal isomorphism g_K^* on H^*.

The notation g_K will be used throughout. The function g_K takes $\mathbb{H} \setminus K$ to the standard domain $\mathbb{H}$, so K no longer appears as a defect of the domain. Thus we call g_K the *mapping-out function* of K. The condition $g_K(z) - z \to 0$ at ∞ which makes g_K unique is sometimes called the *hydrodynamic normalization*. The constant a_K, which we will see later is non-negative, is the *half-plane capacity*.

Proof. Set $D = \{z : -z^{-1} \in H\}$. Then $D \subseteq \mathbb{H}$ is a simply connected domain which is a neighbourhood of 0 in $\mathbb{H}$. Choose a bounded open interval $I \subseteq D^0$ containing 0. By Proposition 4.1, there exists a conformal isomorphism $\phi : D \to \mathbb{H}$ which extends to a reflection-invariant conformal isomorphism ϕ^* on D_I^*, with $\phi^*(0) = 0$ and $\arg(\phi^*)'(0) = 0$. Consider the Taylor expansion of ϕ^* at 0. Since ϕ^* maps I into $\mathbb{R}$, the coefficients must all be real. So, as $z \to 0$, we have

$$\phi^*(z) = az + bz^2 + cz^3 + O(|z|^4)$$

for some $a \in (0, \infty)$ and $b, c \in \mathbb{R}$. Define g_K on H by $g_K(z) = -a\phi(-z^{-1})^{-1} - (b/a)$. It is a straightforward exercise to check that g_K is a conformal isomorphism to $\mathbb{H}$ and that g_K has the claimed expansion at ∞, with $a_K = (b/a)^2 - (c/a)$. In particular, $g_K(z) - z$ is bounded near ∞. Now ϕ^* is a homeomorphism of neighbourhoods of 0, so g_K can only take bounded sets to bounded sets. Hence $g_K(z) - z$ is uniformly bounded on H and, by Proposition 4.2, g_K extends by reflection to a conformal isomorphism on H^*.

Finally, if $g : H \to \mathbb{H}$ is any conformal isomorphism such that $g(z) - z \to 0$ as $|z| \to \infty$, then $f = g \circ g_K^{-1}$ is a conformal automorphism of $\mathbb{H}$ with $f(z) - z \to 0$ as $|z| \to \infty$. Then $f(\infty) = \infty$, so $f(z) = \sigma z + \mu$ for some $\sigma \in (0, \infty)$ and $\mu \in \mathbb{R}$ by Corollary 1.5, and then $f(z) = z$ for all z, showing that $g = g_K$. $\square$

The mapping-out function has a simple form for the half-disc $\bar{\mathbb{D}} \cap \mathbb{H}$ and for the slit $(0, i] = \{iy : y \in (0, 1]\}$:

$$g_{\bar{\mathbb{D}} \cap \mathbb{H}}(z) = z + 1/z, \quad g_{(0,i]}(z) = \sqrt{z^2 + 1} = z + 1/(2z) + O(|z|^{-2}). \tag{6}$$

4.3 Properties of the mapping-out function

The following scaling and translation properties may be deduced from the defining characterization of the mapping-out function. The details are left as an exercise.

Proposition 4.4. *Let K be a compact $\mathbb{H}$-hull. Let $r \in (0, \infty)$ and $x \in \mathbb{R}$. Set*

$$rK = \{rz : z \in K\}, \quad K + x = \{z + x : z \in K\}.$$

Then rK and $K + x$ are compact $\mathbb{H}$-hulls and we have

$$g_{rK}(z) = r g_K(z/r), \quad g_{K+x}(z) = g_K(z - x) + x.$$

Nested compact $\mathbb{H}$-hulls $K_0 \subseteq K$ may be encoded by the composition of mapping-out functions.

Proposition 4.5. *Let K_0 and K_1 be compact $\mathbb{H}$-hulls. Set $K = K_0 \cup g_{K_0}^{-1}(K_1)$. Then K is a compact $\mathbb{H}$-hull K containing K_0 and we have*

$$g_K = g_{K_1} \circ g_{K_0}, \quad a_K = a_{K_0} + a_{K_1}. \tag{7}$$

Moreover we obtain all compact $\mathbb{H}$-hulls K containing K_0 in this way.

Proof. Set $H_0 = \mathbb{H} \backslash K_0$ and $H = \mathbb{H} \backslash K$. We can define a conformal isomorphism $g : H \to \mathbb{H}$ by $g = g_{K_1} \circ g_{K_0}$. In particular H is a simply connected domain. Consider a sequence of points (z_n) in H_0 with $|z_n| \to \infty$. Then $g_{K_0}(z_n)/z_n \to 1$ and $|g_{K_0}(z_n)| \to \infty$. Hence there exists N such that for all $n \geqslant N$ we have $g_{K_0}(z_n) \notin K_1$ and then

$$z_n(g(z_n) - z_n) = z_n(g_{K_1}(g_{K_0}(z_n)) - g_{K_0}(z_n)) + z_n(g_{K_0}(z_n) - z_n) \to a_{K_1} + a_{K_0}.$$

Hence K is bounded and $g = g_K$ and $a_K = a_{K_0} + a_{K_1}$.

On the other hand, suppose K is any compact $\mathbb{H}$-hull containing K_0. Define $K_1 = g_{K_0}(K \setminus K_0)$ and $H_1 = g_{K_0}(\mathbb{H} \setminus K)$. Then $K = K_0 \cup g_{K_0}^{-1}(K_1)$ and $H_1 = \mathbb{H} \setminus K_1$. Also, K_1 is bounded and H_1 is a simply connected domain, so K_1 is a compact $\mathbb{H}$-hull, as required. □

5 Estimates for the mapping-out function

5.1 Boundary estimates

Recall from Section 3.1 the Brownian limit $\hat{B}_{T(H)}$ which is a random variable in the Martin boundary δH. Recall also that g_K extends to a homeomorphism from δH to $\delta\mathbb{H} = \mathbb{R}\cup\{\infty\}$.

Proposition 5.1. *Let $S \subseteq \delta H$ be measurable. Then*

$$\lim_{y\to\infty,\, x/y\to 0} \pi y \mathbb{P}_{x+iy}(\hat{B}_{T(H)} \in S) = \mathrm{Leb}(g_K(S)). \tag{8}$$

Proof. Write $g_K(x+iy) = u+iv$. Then $u/y \to 0$ and $v/y \to 1$ as $y \to \infty$ with $x/y \to 0$. By conformal invariance of Brownian motion, and using the known form (3.2) for the density of harmonic measure in $\mathbb{H}$, we have

$$\mathbb{P}_{x+iy}(\hat{B}_{T(H)} \in S) = \mathbb{P}_{u+iv}(B_{T(\mathbb{H})} \in g_K(S)) = \int_{g_K(S)} \frac{v}{\pi((t-u)^2+v^2)}dt$$

On multipying by πy and letting $y \to \infty$ and $x/y \to 0$ we obtain the desired formula. □

For an interval $(a,b) \subseteq H^0$, we can take $S = (a,b)$ and $x = 0$ in Proposition 5.1 to obtain

$$g_K(b) - g_K(a) = \lim_{y\to\infty} \pi y \mathbb{P}_{iy}(B_{T(H)} \in (a,b)). \tag{9}$$

On the other hand, we can also take $S = \delta H \setminus H^0$ to obtain

$$\lim_{y\to\infty} \pi y \mathbb{P}_{iy}(B_{T(H)} \in K) = \lim_{y\to\infty} \pi y \mathbb{P}_{iy}(B_{T(H)} \notin H^0) = \mathrm{Leb}(\mathbb{R} \setminus g_K(H^0)). \tag{10}$$

Here we used the fact that $\partial H \setminus (K \cup H^0)$ is countable for the first equality.

Proposition 5.2. *Let K be a compact $\mathbb{H}$-hull and let $x \in \mathbb{R}$. Suppose that the interval $[x,\infty)$ does not intersect $\bar{K}$. Then $g_K(x) \geqslant x$. If also $K \subseteq \mathbb{D}$ and $x \in (1,\infty)$, then $g_K(x) \leqslant x + 1/x$.*

Proof. For $b > x$ and $y > \mathrm{rad}(K)$, we have

$$\mathbb{P}_{iy}(B_{T(H)} \in (x,b)) \leqslant \mathbb{P}_{iy}(B_{T(\mathbb{H})} \in (x,b)).$$

Multiply by πy and let $y \to \infty$, using Proposition 5.1, to obtain $g_K(b) - g_K(x) \leqslant b - x$. Subtract b and let $b \to \infty$ to see that $g_K(x) \geqslant x$. If $K \subseteq \mathbb{D}$ and $x \in (1,\infty)$, then also

$$\mathbb{P}_{iy}(B_{T(\mathbb{H}\setminus\bar{\mathbb{D}})} \in (x,b)) \leqslant \mathbb{P}_{iy}(B_{T(H)} \in (x,b)).$$

Multiply by πy and let $y \to \infty$, using Proposition 5.1 again and the known form (6) of the mapping-out function for $\bar{\mathbb{D}} \cap \mathbb{H}$, to obtain

$$(b + 1/b) - (x + 1/x) \leqslant g_K(b) - g_K(x).$$

Then subtract b and let $b \to \infty$ to see that $g_K(x) \leqslant x + 1/x$. □

5.2 Continuity estimate

Define

$$\operatorname{rad}(K) = \inf\{r \geqslant 0 : K \subseteq r\bar{\mathbb{D}} + x \text{ for some } x \in \mathbb{R}\}.$$

Proposition 5.3. *Let K be a compact $\mathbb{H}$-hull. Then*

$$|g_K(z) - z| \leqslant 3\operatorname{rad}(K), \quad z \in H. \tag{11}$$

Proof. By a scaling and translation argument, using Proposition 4.4, it will suffice to consider the case where $K \subseteq \bar{\mathbb{D}}$ and $\operatorname{rad}(K) = 1$. Fix $z \in H$ and consider a complex Brownian motion $(B_t)_{t\geqslant 0}$ starting from z. For $t < T = T(H)$, set $G_t = g_K(B_t)$. By conformal invariance of Brownian motion, G_t converges almost surely as $t \uparrow T$ to a limit $G_T \in \mathbb{R}$. Moreover $G_T \in g_K(H^0)$ if and only if $B_T \in H^0$, and then $G_T = g_K(B_T)$.

Recall that $g_K(z) - z$ is a bounded holomorphic function on H. For $t < T$, set $M_t = g_K(B_t) - B_t = G_t - B_t$. Then $(M_t)_{t<T}$ is a continuous and bounded local martingale, and $M_t \to G_T - B_T$ as $t \uparrow T$. Hence, by optional stopping,

$$g_K(z) - z = \mathbb{E}_z(G_T - B_T). \tag{12}$$

Note that $\{|x| > 1\} \subseteq H^0$ and $\{|x| > 2\} \subseteq g_K(\{|x| > 1\})$. If $|B_T| > 1$, then $B_T \in H^0$, so, by Proposition 5.2, $|G_T - B_T| = |g_K(B_T) - B_T| \leqslant 1/|B_T| \leqslant 1$. On the hand, if $|B_T| \leqslant 1$, then $G_T \notin g_K(\{|x| > 1\})$, so $|G_T| \leqslant 2$. In any case $|G_T - B_T| \leqslant 3$. Hence $|g_K(z) - z| \leqslant 3$. □

5.3 Differentiability estimate

The expansion (5) at ∞ for mapping-out functions states that, for every compact $\mathbb{H}$-hull K, there are constants $C(K) < \infty$ and $R(K) < \infty$ such that,

$$\left| g_K(z) - z - \frac{a_K}{z} \right| \leqslant \frac{C(K)}{|z|^2}, \quad |z| \geqslant R(K).$$

The next result strengthens this estimate, stating that, if $K \subseteq \bar{\mathbb{D}}$, then we can take $C(K) = Ca_K$ and $R(K) = 2$, where $C < \infty$ does not depend on K.

Proposition 5.4. *There is an absolute constant $C < \infty$ with the following properties. For all $r \in (0, \infty)$ and all $\xi \in \mathbb{R}$, for any compact $\mathbb{H}$-hull $K \subseteq r\bar{\mathbb{D}} + \xi$,*

$$\left| g_K(z) - z - \frac{a_K}{z - \xi} \right| \leqslant \frac{Cra_K}{|z - \xi|^2}, \quad |z - \xi| \geqslant 2r. \tag{13}$$

Proof. We shall prove the result in the case $r = 1$ and $\xi = 0$, when $K \subseteq \bar{\mathbb{D}}$. The general case then follows by scaling and translation. Let $D = \mathbb{H} \setminus \bar{\mathbb{D}} = \{z \in \mathbb{H} : |z| > 1\}$. Write $T = T(H)$ and define for $\theta \in [0, \pi]$

$$a(\theta) = \mathbb{E}_{e^{i\theta}}(\operatorname{Im}(B_T)).$$

For $z \in D$, using (12) and then the strong Markov property, we have

$$\operatorname{Im}(z - g_K(z)) = \mathbb{E}_z(\operatorname{Im}(B_T)) = \int_0^\pi h_D(z,\theta)a(\theta)d\theta.$$

Consider the conformal isomorphism $g : D \to \mathbb{H}$ given by $g(z) = z + z^{-1}$. Note that $g(e^{i\theta}) = 2\cos\theta$. Then, for $z \in D$ and $w = g(z)$,

$$h_D(z,\theta) = h_{\mathbb{H}}(w, 2\cos\theta)\frac{d}{d\theta}g(e^{i\theta}) = \operatorname{Im}\left(\frac{1}{2\cos\theta - w}\right)\frac{2\sin\theta}{\pi}$$

by the chain rule. Hence

$$\operatorname{Im}(z - g_K(z)) = \int_0^\pi \operatorname{Im}\left(\frac{1}{2\cos\theta - w}\right)\frac{2\sin\theta}{\pi}a(\theta)d\theta.$$

Set

$$a = \int_0^\pi \frac{2\sin\theta}{\pi}a(\theta)d\theta. \tag{14}$$

Consider the holomorphic function f on $H^* \setminus \{0\}$ given by

$$f(z) = g_K^*(z) - z - a/z,$$

and set $v(z) = \operatorname{Im}(f(z))$. Observe that there is a constant $C < \infty$ such that, for all $|z| \geqslant 3/2$ and $\theta \in [0,\pi]$,

$$\left|\frac{1}{w - 2\cos\theta} - \frac{1}{z}\right| = \frac{|2\cos\theta - z^{-1}|}{|z||z + z^{-1} - 2\cos\theta|} \leqslant \frac{C}{|z|^2}.$$

and hence, for $z \in \mathbb{H}$ with $|z| \geqslant 3/2$,

$$|v(z)| \leqslant \int_0^\pi \left|\frac{1}{w - 2\cos\theta} - \frac{1}{z}\right|\frac{2\sin\theta}{\pi}a(\theta)d\theta \leqslant \frac{Ca}{|z|^2}.$$

Since $v(\bar{z}) = -v(z)$, the same bound holds without the restriction $z \in \mathbb{H}$. Then, for $|z| \geqslant 2$, we can apply Lemma 3.3 in the domain $D_z = \{w \in \mathbb{C} : |w| > (3/4)|z|\}$ to obtain, for a new constant $C < \infty$,

$$\left|\frac{\partial v}{\partial x}(z)\right|, \left|\frac{\partial v}{\partial y}(z)\right| \leqslant \frac{Ca}{|z|^3}.$$

By the Cauchy–Riemann equations, the same bound holds for $|f'(z)|$ for all $|z| \geqslant 3/2$. Now $f(z) \to 0$ as $|z| \to \infty$ so, for $|z| \geqslant 2$ we have,

$$|f(z)| = \left|\int_1^\infty f'(tz)zdt\right| \leqslant \frac{Ca}{|z|^2}\int_1^\infty t^{-3}dt = \frac{Ca}{|z|^2}. \tag{15}$$

Hence $zf(z) \to 0$ as $|z| \to \infty$, so $a = a_K$ and (15) is the desired estimate. □

6 Capacity and half-plane capacity

We discuss two notions of capacity for compact $\mathbb{H}$-hulls and study their properties. The second of these plays a key role in Loewner's theory.

6.1 Capacity from ∞ in $\mathbb{H}$ ($\star$)

Define for a compact $\mathbb{H}$-hull K the *capacity from ∞ in $\mathbb{H}$* by

$$\operatorname{cap}(K) = \lim_{y\to\infty} \pi y \mathbb{P}_{iy}(B_{T(H)} \in K).$$

The existence of this limit was shown in (10). It is clear from the definition that

$$\operatorname{cap}(K) \leqslant \operatorname{cap}(K') \quad \text{whenever} \quad K \subseteq K'.$$

We use (10) together with known properties of mapping-out functions to obtain

$$\operatorname{cap}(\bar{\mathbb{D}} \cap \mathbb{H}) = 4, \quad \operatorname{cap}((0, i]) = 2$$

and, for $r \in (0, \infty)$ and $x \in \mathbb{R}$,

$$\operatorname{cap}(rK) = r \operatorname{cap}(K), \quad \operatorname{cap}(K + x) = \operatorname{cap}(K).$$

Proposition 6.1. *Let K be a compact $\mathbb{H}$-hull such that $\bar{K}$ is connected. Then*

$$\operatorname{rad}(K) \leqslant \operatorname{cap}(K) \leqslant 4 \operatorname{rad}(K).$$

Proof. Set $r = \operatorname{rad}(K)$. Then $K \subseteq r\bar{\mathbb{D}} \cap \mathbb{H} + x$ for some $x \in \mathbb{R}$. So (without using connectedness)

$$\operatorname{cap}(K) \leqslant \operatorname{cap}(r\bar{\mathbb{D}} \cap \mathbb{H} + x) = 4r.$$

By translation and scaling we may assume that $r = 1$ and that there exist $s \in (0, 1]$ and $c \in [0, 1]$ such that $s^2 + c^2 = 1$ and $is \in K$ and either $c \in \bar{K}$ or $-c \in \bar{K}$. Set

$$K_0 = (0, is], \quad \rho(K) = \{-x + iy : x + iy \in K\}, \quad \sigma(K) = K \cup \rho(K).$$

Fix $y \in (1, \infty)$ and consider a complex Brownian motion B starting from iy. Note that B cannot hit $S = K_0 \cup [-c, c]$ without first hitting $\overline{\sigma(K)}$. Hence, by symmetry,

$$\mathbb{P}_{iy}(B_{T(H_0)} \in S) \leqslant 2\mathbb{P}_{iy}(B_{T(H)} \in \bar{K}) = 2\mathbb{P}_{iy}(B_{T(H)} \in K).$$

If $c > 0$ then $g_{K_0}(\pm c) = \pm\sqrt{s^2 + c^2} = \pm 1$, whilst if $c = 0$ then $g_{K_0}(0\pm) = \pm 1$. Hence, by Proposition 5.1, in both cases, on multiplying by πy and letting $y \to \infty$, we obtain

$$2 \leqslant 2 \operatorname{cap}(K).$$

$\square$

Proposition 6.2. *Let A and K be disjoint compact $\mathbb{H}$-hulls. Then*

$$\operatorname{cap}(g_A(K)) \leqslant \operatorname{cap}(K).$$

Proof. Write $g_A(iy) = u + iv$ and recall that $v/y \to 1$ and $u \to 0$ as $y \to \infty$. By conformal invariance of Brownian motion, we have

$$\mathbb{P}_{u+iv}(B \text{ hits } g_A(K) \text{ before } \mathbb{R}) = \mathbb{P}_{iy}(B \text{ hits } K \text{ before } A \cup \mathbb{R}) \leqslant \mathbb{P}_{iy}(B \text{ hits } K \text{ before } \mathbb{R}).$$

Now multiply by πy and let $y \to \infty$, using Proposition 5.1, to obtain the desired inequality. □

6.2 Half-plane capacity

There is a second notion of capacity for a compact $\mathbb{H}$-hull K. Define the *half-plane capacity* by

$$\operatorname{hcap}(K) = \lim_{y \to \infty} y \mathbb{E}_{iy}(\operatorname{Im}(B_{T(H)}).$$

To see that this limit exists, recall from Theorem 4.3 that, as $|z| \to \infty$,

$$z(g_K(z) - z) \to a_K \in \mathbb{R}$$

and, from (12), for all $z \in H$, we have

$$g_K(z) - z = \mathbb{E}_z(G_T - B_T),$$

where $T = T(H)$ and $G_T = g_K(\hat{B}_T) \in \mathbb{R}$. So, taking $z = iy$, we obtain

$$y \mathbb{E}_{iy}(\operatorname{Im}(B_{T(H)})) = -y \operatorname{Im} \mathbb{E}_z(G_T - B_T) = \operatorname{Re}(z(g_K(z) - z)) \to a_K.$$

So the limit not only exists but equals the constant a_K associated to K via its mapping-out function.

From the explicitly known mapping-out functions (6), we deduce that

$$\operatorname{hcap}(\bar{\mathbb{D}} \cap \mathbb{H}) = 1, \quad \operatorname{hcap}(0, i] = 1/2.$$

The following two propositions allow us to relate the half-plane capacities of different compact $\mathbb{H}$-hulls. They follow from Propositions 4.4 and 4.5 and are left as exercises.

Proposition 6.3. *For $r \in (0, \infty)$ and $x \in \mathbb{R}$, we have*

$$\operatorname{hcap}(rK) = r^2 \operatorname{hcap}(K), \quad \operatorname{hcap}(K + x) = \operatorname{hcap}(K).$$

Proposition 6.4. *Let K and K' be compact $\mathbb{H}$-hulls with $K \subseteq K'$. Set $\tilde{K} = g_K(K' \setminus K)$. Then*

$$\operatorname{hcap}(K) \leqslant \operatorname{hcap}(K) + \operatorname{hcap}(\tilde{K}) = \operatorname{hcap}(K').$$

We deduce that, for $K \subseteq r\mathbb{D}$, we have $\mathrm{hcap}(K) \leqslant \mathrm{hcap}(r\bar{\mathbb{D}} \cap \mathbb{H}) = r^2$. Hence, for all compact $\mathbb{H}$-hulls K,

$$\mathrm{hcap}(K) \leqslant \mathrm{rad}(K)^2.$$

Note from the proof of Proposition 3.3 the formula (14)

$$\mathrm{hcap}(K) = \frac{2}{\pi} \int_0^\pi \mathbb{E}_{e^{i\theta}}(\mathrm{Im}(B_{T(H)})) \sin\theta d\theta$$

which shows that $\mathrm{hcap}(K) > 0$ for all non-empty compact $\mathbb{H}$-hulls.

The next result is deeper, relying on Beurling's estimate, which is proved below as Theorem 16.3. It may be considered as a continuity estimate for half-plane capacity.

Proposition 6.5. ($\star$) *Suppose $K \subset K'$ are two compact $\mathbb{H}$-hulls, and that* $\mathrm{dist}(z, \partial K \cup \mathbb{R}) \leqslant$ *ε for all $z \in \partial K'$ and some $\varepsilon > 0$. Then*

$$\mathrm{hcap}(K') \leqslant \mathrm{hcap}(K) + \frac{16}{\pi} \mathrm{rad}(K')^{3/2} \varepsilon^{1/2}.$$

Proof. We reduce to the case where $K' \subseteq \mathbb{D}$ by scaling and translation. Let B be a complex Brownian motion starting from $z \in H'$. Write $T = T(H)$ and $T' = T(H')$ and note that $T \geqslant T'$. By Beurling's estimate, for $z \in \partial K'$ and $r > 0$,

$$\mathbb{P}_z(|B_T - z| \geqslant r) \leqslant \mathbb{P}_z(T \geqslant T(z + r\mathbb{D})) \leqslant 2\sqrt{\varepsilon/r}$$

so, using the strong Markov property at T', for $z = e^{i\theta}$ and $\theta \in (0, \pi)$, we have

$$\mathbb{P}_{e^{i\theta}}(|B_T - B_{T'}| \geqslant r) \leqslant 2\sqrt{\varepsilon/r}.$$

Now $|\mathrm{Im}(B_T) - \mathrm{Im}(B_{T'})| \leqslant |B_T - B_{T'}| \wedge 1$, so

$$\mathbb{E}_{e^{i\theta}}|\mathrm{Im}(B_T) - \mathrm{Im}(B_{T'})| = \int_0^1 \mathbb{P}_{e^{i\theta}}(|B_T - B_{T'}| \geqslant r) dr \leqslant 4\sqrt{\varepsilon}.$$

Then, using (14),

$$\begin{aligned}
\mathrm{hcap}(K') &= \int_0^\pi \mathbb{E}_{e^{i\theta}}(\mathrm{Im}(B_{T'})) \frac{2\sin\theta}{\pi} d\theta \\
&\leqslant \int_0^\pi \mathbb{E}_{e^{i\theta}}(\mathrm{Im}(B_T)) \frac{2\sin\theta}{\pi} d\theta + \int_0^\pi 4\sqrt{\varepsilon} \frac{2\sin\theta}{\pi} d\theta = \mathrm{hcap}(K) + \frac{16}{\pi}\sqrt{\varepsilon}.
\end{aligned}$$

$\square$

7 Chordal Loewner theory I

We establish a one-to-one correspondence between continuous real-valued paths $(\xi_t)_{t\geqslant 0}$ and increasing families $(K_t)_{t\geqslant 0}$ of compact $\mathbb{H}$-hulls having a certain local growth property. The null path $\xi_t \equiv 0$ corresponds to $K_t = (0, 2i\sqrt{t}]$. For smooth paths $(\xi_t)_{t\geqslant 0}$ starting from 0, it is known that $K_t = \{\gamma_s : 0 < s \leqslant t\}$ for some continuous simple path $(\gamma_t)_{t\geqslant 0}$ in $\bar{\mathbb{H}}$ starting from 0 and such that $\gamma_t \in \mathbb{H}$ for all $t > 0$. In the absence of smoothness, the situation can be more complicated, as we shall see later. In this *chordal* version of the theory, the boundary point ∞ plays a special role as the point towards which the hulls evolve. In the alternative *radial* theory, which we will not discuss, an interior point of the domain plays this special role instead.

7.1 Local growth property and Loewner transform

Let $(K_t)_{t\geqslant 0}$ be a family of compact $\mathbb{H}$-hulls. Say that $(K_t)_{t\geqslant 0}$ is *increasing* if K_s is strictly contained in K_t whenever $s < t$. Assume that $(K_t)_{t\geqslant 0}$ is increasing. Set $K_{t+} = \cap_{s>t} K_s$ and, for $s < t$, set $K_{s,t} = g_{K_s}(K_t \setminus K_s)$. Say that $(K_t)_{t\geqslant 0}$ has the *local growth property* if

$$\operatorname{rad}(K_{t,t+h}) \to 0 \quad \text{as } h \downarrow 0 \text{ uniformly on compacts in } t.$$

This is a type of continuity condition for the growth of $(K_t)_{t\geqslant 0}$ but note that $K_t \setminus K_s$ can be large even when $K_{s,t}$ is small. See Figure 4 for an illustration.

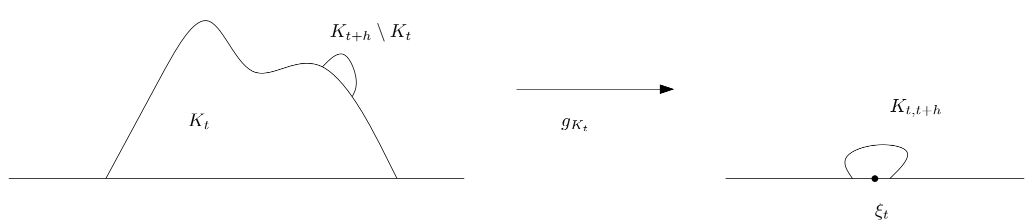


Figure 4: The local growth property and the Loewner transform.

Proposition 7.1. *Let $(K_t)_{t\geqslant 0}$ be an increasing family of compact $\mathbb{H}$-hulls having the local growth property. Then $K_{t+} = K_t$ for all t. Moreover, the map $t \mapsto \operatorname{hcap}(K_t)$ is continuous and strictly increasing on $[0, \infty)$. Moreover, for all $t \geqslant 0$ there is a unique $\xi_t \in \mathbb{R}$ such that $\xi_t \in \overline{K_{t,t+h}}$ for all $h > 0$, and the process $(\xi_t)_{t\geqslant 0}$ is continuous.*

Proof. Set $K_{t,t+} = g_{K_t}(K_{t+} \setminus K_t)$. For all $t \geqslant 0$ and $h > 0$, we have

$$\operatorname{hcap}(K_{t+h}) = \operatorname{hcap}(K_t) + \operatorname{hcap}(K_{t,t+h}).$$

Now $\operatorname{hcap}(K_{t,t+}) \leqslant \operatorname{hcap}(K_{t,t+h}) \leqslant \operatorname{rad}(K_{t,t+h})^2$. Hence, by the local growth property, $t \mapsto \operatorname{hcap}(K_t)$ is continuous and $\operatorname{hcap}(K_{t,t+}) = 0$, so $K_{t,t+} = \emptyset$ and so $K_{t+} = K_t$. On the other hand $K_{t,t+h} \neq \emptyset$ so $\operatorname{hcap}(K_{t,t+h}) > 0$ and so $t \mapsto \operatorname{hcap}(K_t)$ is strictly increasing on $[0, \infty)$.

For fixed $t \geqslant 0$, the sets $\overline{K_{t,t+h}}$ are compact and decreasing in $h > 0$ so, using the local growth property, they have a unique common element $\xi_t \in \mathbb{R}$. For $t \geqslant 0$ and $h > 0$, choose $z \in K_{t+2h} \setminus K_{t+h}$ and set $w = g_{K_t}(z)$ and $w' = g_{K_{t+h}}(z)$. Then $w \in K_{t,t+2h}$ and $w' \in K_{t+h,t+2h}$, with $w' = g_{K_{t,t+h}}(w)$. Hence

$$|\xi_t - w| \leqslant 2\operatorname{rad}(K_{t,t+2h}), \quad |\xi_{t+h} - w'| \leqslant 2\operatorname{rad}(K_{t+h,t+2h}), \quad |w - w'| \leqslant 3\operatorname{rad}(K_{t,t+h})$$

where we used (11) for the last inequality. Hence

$$|\xi_{t+h} - \xi_t| \leqslant 2\operatorname{rad}(K_{t+h,t+2h}) + 3\operatorname{rad}(K_{t,t+h}) + 2\operatorname{rad}(K_{t,t+h}) \to 0$$

as $h \to 0$, uniformly on compacts in t. □

The process $(\xi_t)_{t\geqslant 0}$ is called the *Loewner transform* of $(K_t)_{t\geqslant 0}$. We shall see in the next two subsections that the family of compact $\mathbb{H}$-hulls $(K_t)_{t\geqslant 0}$ can be reconstructed from its Loewner transform.

We shall sometimes be presented with a family of compact $\mathbb{H}$-hulls parametrized not by $[0, \infty)$ but by $[0, T)$ for some $T \in (0, \infty)$. The preceding definitions and results transfer immediately to this case. The following result is left as an exercise.

Proposition 7.2. *Let $T, T' \in (0, \infty]$ and let $\tau : [0, T') \to [0, T)$ be a homeomorphism. Let $(K_t)_{t\in[0,T)}$ be an increasing family of compact $\mathbb{H}$-hulls having the local growth property and having Loewner transform $(\xi_t)_{t\in[0,T)}$. Set $K'_t = K_{\tau(t)}$ and $\xi'_t = \xi_{\tau(t)}$. Then $(K'_t)_{t\in[0,T')}$ is an increasing family of compact $\mathbb{H}$-hulls having the local growth property and having Loewner transform $(\xi'_t)_{t\in[0,T')}$.*

By Proposition 7.1, the map $t \mapsto \operatorname{hcap}(K_t)/2$ is a homeomorphism on $[0, T)$. On choosing τ as the inverse homeomorphism we obtain a family $(K'_t)_{t\in[0,T')}$ such that $\operatorname{hcap}(K'_t) = 2t$ for all t. We say in this case that $(K'_t)_{t\in[0,T')}$ is *parametrized by half-plane capacity.* The 2 is standard in the literature and is present because of a relation with the radial Loewner theory, which we will not discuss.

7.2 Loewner's differential equation

We now come to Loewner's crucial observation: the local growth property implies that the mapping-out functions satisfy a differential equation.

Proposition 7.3. *Let $(K_t)_{t\geqslant 0}$ be an increasing family of compact $\mathbb{H}$-hulls, satisfying the local growth property and parametrized by half-plane capacity, and let $(\xi_t)_{t\geqslant 0}$ be its Loewner transform. Set $g_t = g_{K_t}$ and $\zeta(z) = \inf\{t \geqslant 0 : z \in K_t\}$. Then, for all $z \in \mathbb{H}$, the function $(g_t(z) : t \in [0, \zeta(z)))$ is differentiable, and satisfies* Loewner's differential equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \xi_t}. \tag{16}$$

Moreover, if $\zeta(z) < \infty$, then $g_t(z) - \xi_t \to 0$ as $t \to \zeta(z)$.

Proof. Let $0 \leqslant s < t < \zeta(z)$ and set $z_t = g_t(z)$. Note that $\text{hcap}(K_s) + \text{hcap}(K_{s,t}) = \text{hcap}(K_t)$, so $\text{hcap}(K_{s,t}) = 2(t-s)$. Also, $g_{K_{s,t}}(z_s) = z_t$ and $K_{s,t} \subseteq \xi_s + 2\,\text{rad}(K_{s,t})\bar{\mathbb{D}}$. We apply Propositions 5.3 and 5.4 to the compact $\mathbb{H}$-hull $K_{s,t}$ to obtain

$$|z_t - z_s| \leqslant 6\,\text{rad}(K_{s,t}) \tag{17}$$

and, provided $|z_s - \xi_s| \geqslant 4\,\text{rad}(K_{s,t})$,

$$\left| z_t - z_s - \frac{2(t-s)}{z_s - \xi_s} \right| \leqslant \frac{4C\,\text{rad}(K_{s,t})(t-s)}{|z_s - \xi_s|^2}. \tag{18}$$

We use (17) and the local growth property to see that $(z_t : t \in [0, \zeta(z)))$ is continuous. Then $t \mapsto |z_t - \xi_t|$ is positive and continuous on $[0, \zeta(z))$, and so is locally uniformly positive. Then (18) and the local growth property show that $(z_t : t \in [0, \zeta(z)))$ is differentiable with $\dot{z}_t = 2/(z_t - \xi_t)$. Finally, if $\zeta(z) < \infty$, then for $s < \zeta(z) < t$ we have $z \in K_t \setminus K_s$, so $z_s \in K_{s,t}$, so $|z_s - \xi_s| \leqslant 2\,\text{rad}(K_{s,t})$, and so by the local growth property $|z_s - \xi_s| \to 0$ as $s \to \zeta(z)$. □

7.3 Understanding the Loewner transform

This section is for orientation and aims to develop understanding of how the geometry of a curve $(\gamma_t)_{t \geqslant 0}$ is reflected in the Loewner transform $(\xi_t)_{t \geqslant 0}$ of the hulls $(K_t)_{t \geqslant 0}$ given by $K_t = \gamma((0,t])$. Anticipating Section 8.1, where we shall see that the transform determines the hulls, this also sheds some light on how a given choice of transform affects the geometry of any resulting curve. Fix $\alpha \in (0, \pi/2)$ and take $\gamma(t) = r(t)e^{i\alpha}$, where $r(t)$ is chosen so that $\text{hcap}(K_t) = 2t$. Note that the scaling map $z \mapsto \lambda z$ takes H_t to $H_{\lambda^2 t}$, so the mapping-out functions $g_t = g_{K_t}$ satisfy $g_{\lambda^2 t}(z) = \lambda g_t(z/\lambda)$. Hence, by Loewner's equation, we have $\xi_{\lambda^2 t} = \lambda \xi_t$, so $\xi_t = c_\alpha \sqrt{t}$, where $c_\alpha = \xi_1$. The value of c_α is known, but we shall be content to see that $c_\alpha > 0$. To see this, fix τ so that $\text{rad}(K_\tau) = 1$ and note that, given $\varepsilon > 0$, we can find $b > 1$ such that $g_\tau(b) \leqslant b + \varepsilon$ and $g_\tau(-b) \geqslant -b - \varepsilon$. Write δ^- for the interval of δH_τ from $-b$ to γ_τ and δ^+ for the interval of δH_τ from γ_τ to b. Then, for $y > 1$

$$\mathbb{P}_{iy}(\hat{B}_{T(H_\tau)} \in \delta^-) \geqslant \mathbb{P}_{iy}(\hat{B}_{T(H_\tau)} \in \delta^+).$$

This is left as an exercise. Now multiply by πy and let $y \to \infty$. By Proposition 5.1, we deduce that

$$g_\tau(\gamma_\tau) - g_\tau(-b) \geqslant g_\tau(b) - g_\tau(\gamma_t).$$

Now $g_\tau(\gamma_\tau) = \xi_\tau = c_\alpha \sqrt{\tau}$, so $2c_\alpha\sqrt{\tau} = 2\xi_\tau \geqslant g_\tau(b) + g_\tau(-b) \geqslant 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, this implies that $c_\alpha \geqslant 0$. But we cannot have $c_\alpha = 0$, since this corresponds to the case $\alpha = \pi/2$. In fact, c_α is decreasing in α with $c_\alpha \to \infty$ as $\alpha \to 0$. Note the infinite initial velocity required for the Loewner transform needed to achieve a "turn to the right" with greater angle of turn for greater c_α. For a "turn to the left", we take $\xi_t = -c_\alpha\sqrt{t}$. The term "driving function" is sometimes used for the Loewner transform, which may be thought as referring not only to the fact that it drives Loewner's differential equation (16), but also to the fact that it is, literally, a function which indicates how to "turn the wheel".

8 Chordal Loewner theory II

Loewner's differential equation offers the prospect that we might recover the family of compact $\mathbb{H}$-hulls $(K_t)_{t\geqslant 0}$ from its Loewner transform $(\xi_t)_{t\geqslant 0}$ by solving the equation, indeed that we might construct such a family $(K_t)_{t\geqslant 0}$ starting from any continuous real-valued function $(\xi_t)_{t\geqslant 0}$. We now show this is true.

8.1 Inversion of the Loewner transform

Fix a continuous real-valued function $(\xi_t)_{t\geqslant 0}$, which we call the *driving function*. Define for $t \geqslant 0$ and $z \in \mathbb{C} \setminus \{\xi_t\}$

$$b(t,z) = \frac{2}{z-\xi_t} = \frac{2(\bar{z}-\xi_t)}{|z-\xi_t|^2}.$$

Note that $b(t,.)$ is holomorphic on $\mathbb{C} \setminus \{\xi_t\}$ and, for $|z-\xi_t|, |z'-\xi_t| \geqslant 1/n$,

$$|b(t,z) - b(t,z')| \leqslant 2n^2 |z-z'|.$$

The following proposition is then a straightforward application of general properties of differential equations. For reasons that will become clear later, while we are mainly interested in solving the differential equation in the upper half-plane, it is convenient to solve it in the entire complex plane.

Proposition 8.1. *For all $z \in \mathbb{C} \setminus \{\xi_0\}$, there is a unique $\zeta(z) \in (0,\infty]$ and a unique continuous map $t \mapsto g_t(z) : [0,\zeta(z)) \to \mathbb{C}$ such that, for all $t \in [0,\zeta(z))$, we have $g_t(z) \neq \xi_t$ and*

$$g_t(z) = z + \int_0^t \frac{2}{g_s(z)-\xi_s} ds \tag{19}$$

and such that $|g_t(z)-\xi_t| \to 0$ as $t \to \zeta(z)$ whenever $\zeta(z) < \infty$. Set $\zeta(\xi_0) = 0$ and define $C_t = \{z \in \mathbb{C} : \zeta(z) > t\}$. Then, for all $t \geqslant 0$, C_t is open, and $g_t : C_t \to \mathbb{C}$ is holomorphic.

The process $(g_t(z) : t \in [0,\zeta(z)))$ is the *maximal solution starting from* z, and $\zeta(z)$ is its *lifetime*. Define

$$K_t = \{z \in \mathbb{H} : \zeta(z) \leqslant t\}, \quad H_t = \{z \in \mathbb{H} : \zeta(z) > t\} = \mathbb{H} \setminus K_t.$$

Fix $z \in \mathbb{H}$ and $s \leqslant t < \zeta(z)$, set $y_s = \operatorname{Im} g_s(z)$ and $\delta = \inf_{s\leqslant t} |z_s - \xi_s|$. Then $\delta > 0$ and $\dot{y}_s \geqslant -2y_s/\delta^2$ so $y_t \geqslant e^{-2t/\delta^2} y_0 > 0$. Hence $g_t(H_t) \subseteq \mathbb{H}$. Although we have defined the functions ζ and g_t on $\mathbb{C}$ and C_t respectively, *it is convenient to agree from now on that ζ and g_t refer to the restrictions of these functions to $\mathbb{H}$ and H_t*, except where we make explicit reference to a larger domain. The family of maps $(g_t)_{t\geqslant 0}$ is then called the *Loewner flow (in $\mathbb{H}$) with driving function $(\xi_t)_{t\geqslant 0}$*.

Proposition 8.2. *The family of sets $(K_t)_{t\geqslant 0}$ is an increasing family of compact $\mathbb{H}$-hulls having the local growth property. Moreover $\operatorname{hcap}(K_t) = 2t$ and $g_{K_t} = g_t$ for all t. Moreover the driving function $(\xi_t)_{t\geqslant 0}$ is the Loewner transform of $(K_t)_{t\geqslant 0}$.*

Proof. For $t \geqslant 0$ and $w \in \mathbb{H}$, we have $\mathrm{Im}(b(t,w)) < 0$, so Loewner's differential equation has a unique solution $(w_s : s \in [0,t])$ in $\mathbb{H}$ with given terminal value $w_t = w$. Then $\zeta(w_0) > t$ and $g_t(w_0) = w$ and w_0 is the unique point in $\mathbb{H}$ with these properties. Hence $g_t : H_t \to \mathbb{H}$ is a bijection. We know that g_t is holomorphic by Proposition 8.1, so g_t is a conformal isomorphism. In particular H_t is simply connected.

We next obtain some basic estimates for the Loewner flow. Fix $T \geqslant 0$ and set $r = \sup_{t \leqslant T} |\xi_t - \xi_0| \vee \sqrt{T}$. Fix $R \geqslant 4r$ and take $z \in \mathbb{H}$ with $|z - \xi_0| \geqslant R$. Define

$$\tau = \inf\{t \in [0, \zeta(z)) : |g_t(z) - z| = r\} \wedge T.$$

Then $\tau < \zeta(z)$ and, for all $t \leqslant \tau$,

$$|g_t(z) - \xi_t| = |(g_t(z) - z) + (z - \xi_0) + (\xi_0 - \xi_t)| \geqslant R - 2r$$

and

$$g_t(z) - z = \int_0^t \frac{2}{g_s(z) - \xi_s} ds, \quad z(g_t(z) - z) - 2t = 2\int_0^t \frac{z - g_s(z) + \xi_s}{g_s(z) - \xi_s} ds$$

so

$$|g_t(z) - z| \leqslant \frac{2t}{R - 2r} \leqslant \frac{t}{r}, \quad |z(g_t(z) - z) - 2t| \leqslant \frac{(4r + 2|\xi_0|)t}{R - 2r}.$$

If $\tau < T$, then the first estimate implies that $|g_\tau(z) - z| \leqslant \tau/r < T/r \leqslant r$, a contradiction. Hence $\tau = T$ and then $\zeta(z) > T$ so $z \in H_T$. Since we may choose $R = 4r$, this implies

$$|z - \xi_0| \leqslant 4r \text{ for all } z \in K_T \tag{20}$$

so K_T is bounded and hence is a compact $\mathbb{H}$-hull. On the other hand, by considering the limit $R \to \infty$ in the second estimate, and then letting $T \to \infty$, we see that $z(g_t(z) - z) \to 2t$ as $|z| \to \infty$, for all $t \geqslant 0$. In particular $g_t(z) - z \to 0$ as $|z| \to \infty$, so $g_t = g_{K_t}$ and then $\mathrm{hcap}(K_t) = 2t$ for all t.

It remains to prove the local growth property and identify the Loewner transform. Fix $s \geqslant 0$. Define for $t \geqslant 0$

$$\tilde{\xi}_t = \xi_{s+t}, \quad \tilde{H}_t = g_s(H_{s+t}), \quad \tilde{K}_t = \mathbb{H} \setminus \tilde{H}_t, \quad \tilde{g}_t = g_{s+t} \circ g_s^{-1}.$$

We can differentiate in t to see that $(\tilde{g}_t)_{t \geqslant 0}$ is the Loewner flow driven by $(\tilde{\xi}_t)_{t \geqslant 0}$, $\tilde{H}_t$ is the domain of $\tilde{g}_t$, and $\tilde{K}_t = g_s(K_{s+t} \setminus K_s) = K_{s,s+t}$. The estimate (20) applies to give

$$|z - \xi_s| \leqslant 4\left(\sup_{s \leqslant u \leqslant s+t} |\xi_u - \xi_s| \vee \sqrt{t}\right) \text{ for all } z \in K_{s,s+t}. \tag{21}$$

Hence $(K_t)_{t \geqslant 0}$ has the local growth property and has Loewner transform $(\xi_t)_{t \geqslant 0}$. $\square$

8.2 The Loewner flow on $\mathbb{R}$ characterizes $\bar{K}_t \cap \mathbb{R}$ ($\star$)

By Proposition 4.3, for all $t \geqslant 0$, the map $g_t : H_t \to \mathbb{H}$ extends to a reflection-invariant conformal isomorphism g_t^* on the reflected domain H_t^*. We now show that this is exactly the extended Loewner flow g_t from Proposition 8.1. Later analysis of properties of SLE relies on this property, while the fact that requires proof has sometimes been overlooked.

For $z \in \mathbb{C}$, define

$$\zeta^*(z) = \inf\{t \geqslant 0 : z \notin H_t^*\}.$$

Proposition 8.3. *We have $\zeta^* = \zeta$ on $\mathbb{C}$. Moreover, $H_t^* = C_t$ and $g_t^* = g_t$ on C_t for all $t > 0$.*

Proof. By taking complex conjugates in (19) and using uniqueness we see that $\zeta(\bar{z}) = \zeta(z)$ on $\mathbb{C}$ and $g_t(\bar{z}) = \overline{g_t(z)}$ for all $z \in \mathbb{C}$ and all $t \in [0, \zeta(z))$. In particular C_t is invariant under conjugation for all t, and $g_t : C_t \to \mathbb{C}$ is a holomorphic extension by reflection of its restriction to H_t for all t. Hence $C_t \subseteq H_t^*$ and g_t is the restriction of g_t^* to C_t for all t.

It remains to show for $t > 0$ and $x \in H_t^0 = H_t^* \cap \mathbb{R}$ that $\zeta(x) > t$. Note first that, for $z \in H_t$ and $r < s \leqslant t$, we have

$$|g_r^*(z) - g_s^*(z)| \leqslant 3\,\mathrm{rad}(K_{r,s}) \tag{22}$$

and this estimate extends to H_t^0 by continuity. We will show further that for $x \in H_t^0$

$$\inf_{s \leqslant t} |g_s^*(x) - \xi_s| > 0.$$

This then allows us to pass to the limit $z \to x$ with $z \in H_t$ in (19), to see that $(g_s^*(x) : s \leqslant t)$ satisfies (19), so $\zeta(x) > t$.

Now, for $x \in H_t^0$ and $s < t$, we have $g_s^*(x) \neq \xi_s$. To see this, note that $x \in H_s^0$ so g_s^* is conformal at x, and there is a sequence (w_n) in K_t such that $g_s^*(w_n) \to \xi_s$; then $g_s^*(x) = \xi_s$ would imply $w_n \to x$, which is impossible. The function $s \mapsto |g_s^*(x) - \xi_s|$ is thus continuous on $[0, t]$ and positive on $[0, t)$. It remains to show that it is also positive at t.

Write I for the interval of H_t^0 containing x. Then $g_t^*(I)$ is an open interval containing $g_t^*(x)$. Consider the intervals

$$J_s = \cap_{r \in [s,t]} g_r^*(I), \quad s < t.$$

For s sufficiently close to t, by (22), J_s contains a neighbourhood of $g_t^*(x)$. Hence, if $g_t^*(x) = \xi_t$, then for some $s < t$, we would have $\xi_s \in J_s$, so $\xi_s = g_s^*(y)$ for some $y \in H_t^0$, which we have shown is impossible. □

An immediate corollary is the following characterization of the set of limit points of K_t in $\mathbb{R}$ in terms of the lifetime ζ of the Loewner flow on $\mathbb{R}$.

Proposition 8.4. *For all $x \in \mathbb{R}$ and all $t > 0$, we have*

$$x \in \bar{K}_t \quad \textit{if and only if} \quad \zeta(x) \leqslant t. \tag{23}$$

8.3 Loewner–Kufarev theorem

Write $\mathcal{K}$ for the set of all compact $\mathbb{H}$-hulls. Fix a metric d of uniform convergence on compacts for $C(\mathbb{H}, \mathbb{H})$. We make $\mathcal{K}$ into a metric space using the *Carathéodory metric*

$$d_{\mathcal{K}}(K_1, K_2) = d(g_{K_1}^{-1}, g_{K_2}^{-1}).$$

Write $\mathcal{L}$ for the set of increasing families of compact $\mathbb{H}$-hulls $(K_t)_{t\geqslant 0}$ having the local growth property and such that $\text{hcap}(K_t) = 2t$ for all t. Then $\mathcal{L} \subseteq C([0,\infty), \mathcal{K})$. We fix on $C([0,\infty), \mathcal{K})$ a metric of uniform convergence on compact time intervals.

Theorem 8.5. *There is a bi-adapted homeomorphism $L : C([0,\infty), \mathbb{R}) \to \mathcal{L}$ given by*

$$L((\xi_t)_{t\geqslant 0}) = (K_t)_{t\geqslant 0}, \quad K_t = \{z \in \mathbb{H} : \zeta(z) \leqslant t\}$$

where $\zeta(z)$ is the lifetime of the maximal solution to Loewner's differential equation

$$\dot{z}_t = 2/(z_t - \xi_t)$$

starting from z. Moreover,

$$\bar{K}_t \cap \mathbb{R} = \{x \in \mathbb{R} : \zeta(x) \leqslant t\}$$

where $\zeta(x)$ is the lifetime of the maximal solution to $\dot{x}_t = 2/(x_t - \xi_t)$ starting from x.

Moreover $(\xi_t)_{t\geqslant 0}$ is then the Loewner transform of $(K_t)_{t\geqslant 0}$, given by

$$\{\xi_t\} = \cap_{s>t} \overline{K_{t,s}}, \quad K_{t,s} = g_{K_t}(K_s \setminus K_t) \tag{24}$$

where g_{K_t} is the mapping-out function for K_t.

We call L the *Loewner map*. The proof that L and its inverse are continuous and adapted is left as an exercise. The rest of the theorem recapitulates the results of the preceding two sections.

9 Schramm–Loewner evolutions

We review the arguments which led Schramm to use a Brownian motion as the driving function in Loewner's theory. Then we state the fundamental result of Rohde and Schramm that associates to the resulting family of compact $\mathbb{H}$-hulls a unique continuous path.

9.1 Schramm's observation

We say that a random variable $(K_t)_{t\geqslant 0}$ in $\mathcal{L}$ is a *Schramm–Loewner evolution*[7] if its Loewner transform is a Brownian motion of some diffusivity $\kappa \in [0,\infty)$. We will refer to such a random family of compact $\mathbb{H}$-hulls as an *SLE*(κ). The Loewner–Kufarev theorem allows us to construct SLE(κ) as $K_t = \{z \in \mathbb{H} : \zeta(z) \leqslant t\}$, where where $\zeta(z)$ is the lifetime of the maximal solution to Loewner's differential equation

$$\dot{z}_t = 2/(z_t - \xi_t)$$

starting from z, and where $(\xi_t)_{t\geqslant 0}$ is a Brownian motion of diffusivity κ.

Schramm's revolutionary observation was that these processes offered the unique possible scaling limits for a range of lattice-based planar random systems at criticality, such as loop-erased random walk, Ising model, percolation and self-avoiding walk. Such limits had been conjectured but without a candidate for the limit object. Any scaling limit is scale invariant. In fact it was widely conjectured that there would be limit objects, associated to some class of planar domains, with a stronger property of invariance under conformal maps. Moreover, the local determination of certain paths in the lattice models suggested a form of 'domain Markov property'.

There is a natural scaling map on $\mathcal{L}$. For $\lambda \in (0,\infty)$ and $(K_t)_{t\geqslant 0} \in \mathcal{L}$, define $K_t^\lambda = \lambda K_{\lambda^{-2}t}$. Recall that $\operatorname{hcap}(\lambda K_t) = \lambda^2 \operatorname{hcap}(K_t)$. We have rescaled time so that $(K_t^\lambda)_{t\geqslant 0} \in \mathcal{L}$. We say that a random variable $(K_t)_{t\geqslant 0}$ in $\mathcal{L}$ is *scale invariant* if $(K_t^\lambda)_{t\geqslant 0}$ has the same distribution as $(K_t)_{t\geqslant 0}$ for all $\lambda \in (0,\infty)$.

There is also a natural time-shift map on $\mathcal{L}$. For $s \in [0,\infty)$ and $(K_t)_{t\geqslant 0} \in \mathcal{L}$, define $K_t^{(s)} = g_{K_s}(K_{s+t} \setminus K_s) - \xi_s$. Then $(K_t^{(s)})_{t\geqslant 0} \in \mathcal{L}$. We say that a random variable $(K_t)_{t\geqslant 0}$ in $\mathcal{L}$ has the *domain Markov property* if $(K_t^{(s)})_{t\geqslant 0}$ has the same distribution as $(K_t)_{t\geqslant 0}$ and is independent of $\mathcal{F}_s = \sigma(\xi_r : r \leqslant s)$ for all $s \in [0,\infty)$.

Theorem 9.1. *Let $(K_t)_{t\geqslant 0}$ be a random variable in $\mathcal{L}$. Then $(K_t)_{t\geqslant 0}$ is an SLE if and only if $(K_t)_{t\geqslant 0}$ is scale invariant and has the domain Markov property.*

Proof. Write $(\xi_t)_{t\geqslant 0}$ for the Loewner transform of $(K_t)_{t\geqslant 0}$ and note that $(\xi_t)_{t\geqslant 0}$ is continuous. For $\lambda \in (0,\infty)$ and $s \in [0,\infty)$, define $\xi_t^\lambda = \lambda \xi_{\lambda^{-2}t}$ and $\xi_t^{(s)} = \xi_{s+t} - \xi_s$. Then $(K_t^\lambda)_{t\geqslant 0}$ has Loewner transform $(\xi_t^\lambda)_{t\geqslant 0}$ and $(K_t^{(s)})_{t\geqslant 0}$ has Loewner transform $(\xi_t^{(s)})_{t\geqslant 0}$. Hence $(K_t)_{t\geqslant 0}$ has the domain Markov property if and only if $(\xi_t)_{t\geqslant 0}$ has stationary independent increments. Also $(K_t)_{t\geqslant 0}$ is scale invariant if and only if the law of $(\xi_t)_{t\geqslant 0}$ is

[7] In Schramm's papers, SLE stood for *stochastic Loewner evolution*. As usual, our default assumption is that Brownian motion starts at 0.

invariant under Brownian scaling. By the Lévy–Khinchin Theorem[8] , $(\xi_t)_{t\geqslant 0}$ has both these properties if and only if it is a Brownian motion of some diffusivity $\kappa \in [0,\infty)$, that is to say, if and only if $(K_t)_{t\geqslant 0}$ is an SLE. □

9.2 Rohde–Schramm theorem

A continuous path $(\gamma_t)_{t\geqslant 0}$ in $\bar{\mathbb{H}}$ is said to generate an increasing family of compact $\mathbb{H}$-hulls $(K_t)_{t\geqslant 0}$ if $H_t = \mathbb{H}\setminus K_t$ is the unbounded component of $\mathbb{H}\setminus\gamma[0,t]$ for all t, where $\gamma[0,t] = \{\gamma_s : s \in [0,t]\}$. Rohde and Schramm proved the following fundamental and hard result, except for the case $\kappa = 8$, which was then added by Lawler, Schramm and Werner. We refer to the original papers [4, 5] for the proof.

Theorem 9.2. *Let $(K_t)_{t\geqslant 0}$ be an SLE(κ) for some $\kappa \in [0,\infty)$. Write $(g_t)_{t\geqslant 0}$ and $(\xi_t)_{t\geqslant 0}$ for the associated Loewner flow and transform. The map $g_t^{-1} : \mathbb{H} \to H_t$ extends continuously to $\bar{\mathbb{H}}$ for all $t \geqslant 0$, almost surely. Moreover, if we set $\gamma_t = g_t^{-1}(\xi_t)$, then $(\gamma_t)_{t\geqslant 0}$ is continuous and generates $(K_t)_{t\geqslant 0}$, almost surely.*

We call $(\gamma_t)_{t\geqslant 0}$ an *SLE(κ) path*, or simply an SLE(κ), allowing the notation to signal that we mean the path rather than the hulls.

9.3 SLE in a two-pointed domain

By a *two-pointed domain* we mean a triple $\boldsymbol{D} = (D, z_0, z_\infty)$, where D is a proper simply connected planar domain and z_0 and z_∞ are distinct points in the Martin boundary δD. Write $\mathcal{D}$ for the set of all two-pointed domains. By a *conformal isomorphism of two-pointed domains* $(D, z_0, z_\infty) \to (D', z_0', z_\infty')$, we mean a conformal isomorphism $\phi : D \to D'$ such that $\phi(z_0) = z_0'$ and $\phi(z_\infty) = z_\infty'$. We call any conformal isomorphism $\sigma : \boldsymbol{D} \to (\mathbb{H}, 0, \infty)$ a *scale* for $\boldsymbol{D}$. By Corollary 1.6, such a scale σ exists for all $\boldsymbol{D} \in \mathcal{D}$. Moreover, for all $\lambda \in (0,\infty)$, the map $z \mapsto \lambda\sigma(z)$ is also a scale for $\boldsymbol{D}$ and, by Corollary 1.5, these are all the scales for $\boldsymbol{D}$.

Fix $\boldsymbol{D} = (D, z_0, z_\infty) \in \mathcal{D}$ and a scale σ for $\boldsymbol{D}$. We call a subset $K \subseteq D$ a *$\boldsymbol{D}$-hull* if $D \setminus K$ is a simply connected neighbourhood of z_∞ in D. Write $\mathcal{K}(\boldsymbol{D})$ for the set of all $\boldsymbol{D}$-hulls. Note that $\mathcal{K}(\mathbb{H}, 0, \infty)$ is simply the set $\mathcal{K}$ of compact $\mathbb{H}$-hulls. The Carathéodory topology on $\mathcal{K}$ is scale invariant. For each choice of scale σ, the map $K \mapsto \sigma(K)$ is a bijection $\mathcal{K}(\boldsymbol{D}) \to \mathcal{K}$. We use this bijection to define the Carathéodory topology on $\mathcal{K}(\boldsymbol{D})$, which is then independent of the choice of scale. Similarly, we extend to increasing families of $\boldsymbol{D}$-hulls the notion of the local growth property. Write $\mathcal{L}(\boldsymbol{D}, \sigma)$ for the set of increasing families $(K_t)_{t\geqslant 0}$ of $\boldsymbol{D}$-hulls having the local growth property and such that $\mathrm{hcap}(\sigma(K_t)) = 2t$ for all t. The set $\mathcal{L}$ defined in Section 8.3 corresponds to the case $\boldsymbol{D} = (\mathbb{H}, 0, \infty)$ and $\sigma(z) = z$, to which we default unless $\boldsymbol{D} \in \mathcal{D}$ and a scale σ on $\boldsymbol{D}$ are explicitly mentioned. We use on $\mathcal{L}(\boldsymbol{D}, \sigma)$ the topology of uniform convergence on compact time intervals.

[8] From the Lévy-Khinchin representation, the only continuous Lévy processes are scaled Brownian motions with constant drift, and the scaling invariance forces the drift to vanish.

For $\kappa \in [0,\infty)$, we say that a random variable $(K_t)_{t\geqslant 0}$ in $\mathcal{L}(\boldsymbol{D},\sigma)$ is an *SLE*(κ) *in* $\boldsymbol{D}$ *of scale* σ if the Loewner transform $(\xi_t)_{t\geqslant 0}$ of$(\sigma(K_t))_{t\geqslant 0}$ is a Brownian motion of diffusivity κ. We will write in this context $D_t = D \setminus K_t$ and $g_t = g_{\sigma(K_t)} \circ \sigma$ and $\mathcal{F}_t = \sigma(\xi_s : s \leqslant t)$. The following result is a straightforward translation of the scaling property of SLE. The proof is left as an exercise.

Proposition 9.3 (Conformal invariance of SLE)**.** *Let* $\phi : \boldsymbol{D} \to \boldsymbol{D}'$ *be a conformal isomorphism of two-pointed domains. Fix scales* σ *for* $\boldsymbol{D}$ *and* σ' *for* $\boldsymbol{D}'$ *and set* $\lambda = \sigma' \circ \phi \circ \sigma^{-1} \in (0,\infty)$. *Let* $(K_t)_{t\geqslant 0}$ *be an SLE*(κ) *in* $\boldsymbol{D}$ *of scale* σ. *Set* $K'_t = \phi(K_{\lambda^{-2}t})$. *Then* $(K'_t)_{t\geqslant 0}$ *is an SLE*(κ) *in* $\boldsymbol{D}'$ *of scale* σ'.

In particular we see that any property of an SLE(κ) which is invariant under linear change of time-scale is also insensitive to the choice of scale σ. The domain Markov property can be put in a more striking form in the present context. The proof is again left as an exercise. You will need to use the strong Markov property of Brownian motion.

Proposition 9.4 (Domain Markov property of SLE)**.** *Let* $(K_t)_{t\geqslant 0}$ *be an SLE*(κ) *in* (D, z_0, z_∞) *of scale* σ *and let* T *be a finite stopping time. Set* $\tilde{K}_t = K_{T+t} \setminus K_T$. *Define* $\sigma_T : D_T \to \mathbb{H}$ *by* $\sigma_T(z) = g_T(z) - \xi_T$, *and define* $z_T \in \delta D_T$ *by* $z_T = g_T^{-1}(\xi_T)$. *Then* $(D_T, z_T, z_\infty) \in \mathcal{D}$ *and* σ_T *is a scale for* (D_T, z_T, z_∞). *Moreover, conditional on* $\mathcal{F}_T$, $(\tilde{K}_t)_{t\geqslant 0}$ *is an SLE*(κ) *in* (D_T, z_T, z_∞) *of scale* σ_T.

10 Bessel flow and hitting probabilities for SLE

We begin an analysis of the properties of SLE. This is achieved not directly but by establishing first some properties of the associated Loewner flow – an approach which will recur below. Our goal in this section is to determine for which parameter values the SLE path hits the boundary of its domain and, when it does, to calculate some associated probabilities.

10.1 Bessel flow

Consider the Loewner flow $(g_t(x) : t \in [0, \zeta(x)), x \in \mathbb{R} \setminus \{0\})$ on $\mathbb{R}$ associated to SLE(κ). Recall that the Loewner transform $(\xi_t)_{t \geqslant 0}$ is a Brownian motion of diffusivity κ. Recall also that, for each $x \in \mathbb{R} \setminus \{0\}$, for all $t \in [0, \zeta(x))$, we have $g_t(x) \neq \xi_t$ and

$$g_t(x) = x + \int_0^t \frac{2}{g_s(x) - \xi_s} ds$$

with $g_t(x) - \xi_t \to 0$ as $t \to \zeta(x)$ whenever $\zeta(x) < \infty$. Set

$$a = \frac{2}{\kappa}, \quad B_t = -\frac{\xi_t}{\sqrt{\kappa}}, \quad \tau(x) = \zeta(x\sqrt{\kappa})$$

and for $t \in [0, \tau(x))$ set

$$X_t(x) = \frac{g_t(x\sqrt{\kappa}) - \xi_t}{\sqrt{\kappa}}.$$

Then $(B_t)_{t \geqslant 0}$ is a standard Brownian motion starting from 0. Moreover, for all $x \in \mathbb{R} \setminus \{0\}$ and $t \in [0, \tau(x))$, we have $X_t(x) \neq 0$ and

$$X_t(x) = x + B_t + \int_0^t \frac{a}{X_s(x)} ds \tag{25}$$

with $X_t(x) \to 0$ as $t \to \tau(x)$ whenever $\tau(x) < \infty$. This is the *Bessel flow of parameter a driven by $(B_t)_{t \geqslant 0}$*.

We note two simple properties. First, by considering uniqueness of solutions in reversed time, we obtain the following monotonicity property: for $x, y \in (0, \infty)$ with $x < y$, we have $\tau(x) \leqslant \tau(y)$ and $X_t(x) < X_t(y)$ for all $t < \tau(x)$. Second, there is a scaling property. Fix $\lambda \in (0, \infty)$ and set

$$\tilde{B}_t = \lambda B_{\lambda^{-2}t}, \quad \tilde{\tau}(x) = \lambda^2 \tau(\lambda^{-1}x), \quad \tilde{X}_t(x) = \lambda X_{\lambda^{-2}t}(\lambda^{-1}x).$$

Then $(\tilde{B}_t)_{t \geqslant 0}$ is a Brownian motion. Moreover the family of processes $(\tilde{X}_t(x) : t \in [0, \tilde{\tau}(x)), x \in \mathbb{R} \setminus \{0\})$ is the Bessel flow of parameter a driven by $(\tilde{B}_t)_{t \geqslant 0}$, and hence has the same distribution as $(X_t(x) : t \in [0, \tau(x)), x \in \mathbb{R} \setminus \{0\})$.

Proposition 10.1. *Let $x, y \in (0, \infty)$ with $x < y$. Then*

(a) *for $a \in (0, 1/4]$, we have*
$$\mathbb{P}(\tau(x) < \tau(y) < \infty) = 1;$$

(b) *for $a \in (1/4, 1/2)$, we have*
$$\mathbb{P}(\tau(x) < \infty) = 1, \quad \mathbb{P}(\tau(x) < \tau(y)) = \phi\left(\frac{y - x}{y}\right)$$
where ϕ is given by
$$\phi(\theta) \propto \int_0^\theta \frac{du}{u^{2-4a}(1-u)^{2a}}, \quad \phi(1) = 1; \tag{26}$$

(c) *for $a \in [1/2, \infty)$, we have*
$$\mathbb{P}(\tau(x) < \infty) = 0$$
and moreover, for $a \in (1/2, \infty)$, we have $X_t(x) \to \infty$ as $t \to \infty$ almost surely.

Proof. Fix $x > 0$ and write $X_t = X_t(x)$ and $\tau = \tau(x)$. For $r \in (0, \infty)$ define a stopping time
$$T(r) = \inf\{t \in [0, \tau) : X_t = r\}.$$
Fix $r, R \in (0, \infty)$ and assume that $0 < r < x < R$. Write $S = T(r) \wedge T(R)$. Note that $T(r) < \tau$ on $\{\tau < \infty\}$. Also, $X_t \geqslant B_t + x$ for all $t < \tau$, so $T(R) < \infty$ almost surely on $\{\tau = \infty\}$. In particular, $S < \infty$ almost surely.

Assume for now that $a \neq 1/2$. Set $M_t = X_t^{1-2a}$ for $t < \tau$. Note that M^S is uniformly bounded. By Itô's formula
$$dM_t = (1 - 2a)X_t^{-2a} dX_t - a(1 - 2a)X_t^{-2a-1} dt = (1 - 2a)X_t^{-2a} dB_t.$$

Hence M^S is a bounded martingale and by optional stopping
$$x^{1-2a} = M_0 = \mathbb{E}(M_S) = r^{1-2a}\mathbb{P}(X_S = r) + R^{1-2a}\mathbb{P}(X_S = R). \tag{27}$$

Note that as $r \downarrow 0$ we have $\{X_S = R\} \uparrow \{T(R) < \tau\}$ and so $\mathbb{P}(X_S = R) \to \mathbb{P}(T(R) < \tau)$. Similarly, $\mathbb{P}(X_S = r) \to \mathbb{P}(T(r) < \infty)$ as $R \to \infty$. For $a \in (0, 1/2)$, we can let $r \to 0$ in (27) to obtain
$$\mathbb{P}(T(R) < \tau) = (x/R)^{1-2a}.$$
Then, letting $R \to \infty$, we deduce that $\mathbb{P}(\tau = \infty) = 0$. For $a \in (1/2, \infty)$, we consider the limit $r \to 0$. Then (27) forces $\mathbb{P}(X_S = r) \to 0$, so $\mathbb{P}(T(R) < \tau) = 1$ for all R and hence $\mathbb{P}(\tau = \infty) = 1$. Now M is positive and, as a continuous local martingale, M is also a time-change of Brownian motion. Hence $M_t = X_t^{1-2a}$ must converge almost surely as $t \to \infty$, and the total quadratic variation $[M]_\infty = (2a - 1)^2 \int_0^\infty X_t^{-4a} dt$ must be finite almost surely. This forces $X_t \to \infty$ as $t \to \infty$ almost surely.

In the case $a = 1/2$, we instead set $M_t = \log X_t$ and argue as above to obtain

$$\log x = \mathbb{P}(X_S = r)\log r + \mathbb{P}(X_S = R)\log R.$$

The same argument as for $a \in (1/2, \infty)$ can then be used to see that $\mathbb{P}(\tau = \infty) = 1$.

Assume from now on that $a \in (0, 1/2)$. It remains to show for $0 < x < y$ that

$$\mathbb{P}(\tau < \tau(y)) = \begin{cases} 1, & \text{if } a \leqslant 1/4 \\ \phi(\frac{y-x}{y}), & \text{if } a > 1/4. \end{cases}$$

Define for $\theta \in [0, 1]$

$$\chi(\theta) = \int_\theta^1 \frac{du}{u^{2-4a}(1-u)^{2a}}.$$

Note that χ is continuous on $[0, 1]$ as a map into $[0, \infty]$, with $\chi(0) < \infty$ for $a \in (1/4, 1/2)$ and $\chi(0) = \infty$ for $a \in (0, 1/4]$. Note also that χ is C^2 on $(0, 1)$, with

$$\chi''(\theta) + 2\left(\frac{1-2a}{\theta} - \frac{a}{1-\theta}\right)\chi'(\theta) = 0.$$

Fix $y > x$ and write $Y_t = X_t(y)$. For $t < \tau$, define $R_t = Y_t - X_t$, $\theta_t = R_t/Y_t$ and $N_t = \chi(\theta_t)$. By Itô's formula

$$dR_t = -\frac{aR_t dt}{X_t Y_t}, \quad d\theta_t = \left(\frac{\theta_t}{Y_t}\right)^2 \left(\frac{1-2a}{\theta_t} - \frac{a}{1-\theta_t}\right) dt - \frac{\theta_t}{Y_t} dB_t$$

so

$$dN_t = \chi'(\theta_t)d\theta_t + \frac{1}{2}\chi''(\theta_t)d\theta_t d\theta_t = -\frac{\chi'(\theta_t)\theta_t dB_t}{Y_t}.$$

Hence $(N_t : t < \tau)$ is a local martingale. Now N is non-negative and is a time-change of Brownian motion, so N_t must converge to some limit as $t \to \tau$. Since χ is strictly decreasing, it follows that θ_t converges to some limit θ_τ as $t \to \tau$.

If $\tau < \tau(y)$, then $\theta_\tau = 1$ so $N_\tau = 0$. On the other hand we claim that if $\tau = \tau(y)$ then $\theta_\tau = 0$ almost surely. Indeed, note that we necessarily have $[N]_\tau < \infty$ almost surely, and

$$[N]_t = \int_0^t \frac{\chi'(\theta_s)^2\theta_s^2}{Y_s^2} ds.$$

If $\theta_\tau > 0$ then it follows that

$$\int_0^{\tau(y)} \frac{ds}{Y_s^2} < \infty.$$

Consider the random variables

$$A(x) = \int_0^\tau \frac{1}{X_t^2} dt, \quad A_n(x) = \int_{T(2^{-n+1}x)}^{T(2^{-n}x)} \frac{1}{X_t^2} dt, \quad n \geqslant 1.$$

By the strong Markov property (of the driving Brownian motion), the random variables $(A_n(x) : n \in \mathbb{N})$ are independent. By the scaling property, they all have the same distribution. Hence, since $A_1(x) > 0$ almost surely, we must have $A(x) = \sum_n A_n(x) = \infty$ almost surely. We conclude that if $\tau = \tau_y$ then $\theta_\tau = 0$.

In the case $a \in (0, 1/4]$, $\tau = \tau_y$ would thus imply that $N_t = \chi(\theta_t) \to \infty$ as $t \uparrow \tau$, a contradiction, so $\mathbb{P}(\tau < \tau(y)) = 1$. On the other hand, for $a \in (1/4, 1/2)$, the process N^τ is a bounded martingale so by optional stopping

$$\chi\left(\frac{y-x}{y}\right) = N_0 = \mathbb{E}(N_\tau) = \chi(0)\mathbb{P}(\tau = \tau(y)).$$

□

A variation of the calculation for $\mathbb{P}(\tau(x) < \tau(y))$ allows us to compute $\mathbb{P}(\tau(x) < \tau(-y))$.

Proposition 10.2. *Let $x, y \in (0, \infty)$. Then for $a \in (0, 1/2)$ we have*

$$\mathbb{P}(\tau(x) < \tau(-y)) = \psi\left(\frac{y}{x+y}\right)$$

where ψ is given by

$$\psi(\theta) \propto \int_0^\theta \frac{du}{u^{2a}(1-u)^{2a}}, \quad \psi(1) = 1. \tag{28}$$

Proof. Note that ψ is continuous and increasing on $[0, 1]$ with $\psi(0) = 0$ and $\psi(1) = 1$. Also ψ is C^2 on $(0, 1)$ with

$$\psi''(\theta) + 2a\left(\frac{1}{\theta} - \frac{1}{1-\theta}\right)\psi'(\theta) = 0.$$

Write $X_t = X_t(x)$ and $Y_t = -X_t(-y)$ and set $T = \tau(x) \wedge \tau(-y)$. For $t \leqslant T$ set $R_t = X_t + Y_t$ and $\theta_t = Y_t/R_t$. Define a process $Q = (Q_t)_{t \geqslant 0}$ by setting $Q_t = \psi(\theta_{T \wedge t})$. Then Q is continuous and uniformly bounded. Note that $\theta_T = 1$ if $\tau(x) < \tau(-y)$ and $\theta_T = 0$ if $\tau(-y) < \tau(x)$, and that $Q_T = \theta_T$. By Itô's formula, for $t \leqslant T$,

$$dR_t = \frac{aR_t}{X_t Y_t} dt, \quad d\theta_t = \frac{a}{R_t^2}\left(\frac{1}{\theta_t} - \frac{1}{1-\theta_t}\right) dt - \frac{dB_t}{R_t}$$

so

$$dQ_t = \psi'(\theta_t) d\theta_t + \frac{1}{2}\psi''(\theta_t) d\theta_t d\theta_t = -\frac{\psi'(\theta_t) dB_t}{R_t}.$$

Hence Q is a bounded martingale. By optional stopping

$$\mathbb{P}(\tau(x) < \tau(-y)) = \mathbb{P}(\theta_T = 1) = \mathbb{E}(Q_T) = Q_0 = \psi(\theta_0) = \psi\left(\frac{y}{x+y}\right).$$

□

10.2 Hitting probabilities for SLE(κ) on the real line

We translate the results for the Bessel flow back in terms of the path γ of an SLE(κ).

Proposition 10.3. *Let γ be an SLE(κ). Then*

(a) *for $\kappa \in (0, 4]$, we have $\gamma[0, \infty) \cap \mathbb{R} = \{0\}$ almost surely;*

(b) *for $\kappa \in (4, 8)$ and all $x, y \in (0, \infty)$, γ hits both $[x, \infty)$ and $(-\infty, -y]$ almost surely, and*

$$\mathbb{P}(\gamma \text{ hits } [x, x+y)) = \phi\left(\frac{y}{x+y}\right), \quad \mathbb{P}(\gamma \text{ hits } [x, \infty) \text{ before } (-\infty, -y]) = \psi\left(\frac{y}{x+y}\right)$$

where ϕ and ψ are given by (26) and (28) respectively;

(c) *for $\kappa \in [8, \infty)$, we have $\mathbb{R} \subseteq \gamma[0, \infty)$ almost surely.*

Proof. Fix $x, y \in (0, \infty)$ and $t > 0$. If $\gamma[0, t] \cap [x, \infty) = \emptyset$ then by compactness there is a neighbourhood of $[x, \infty)$ in $\mathbb{H}$ disjoint from $\gamma[0, t]$ which is then contained in H_t, so $x \notin \bar{K}_t$, and so $\zeta(x) > t$ by Proposition 8.4. On the other hand, if $\gamma_s \in [x, \infty)$ for some $s \in [0, t]$, then $\gamma_s \in \bar{K}_t$ so $\zeta(x) \leqslant \zeta(\gamma_s) \leqslant t$, also by Proposition 8.4. Hence

$$\{\gamma[0, t] \text{ hits } [x, \infty)\} = \{\zeta(x) \leqslant t\}, \quad \{\gamma \text{ hits } [x, x+y)\} = \{\zeta(x) < \zeta(x+y)\}.$$

Recall that $\zeta(x) = \tau(x/\sqrt{\kappa})$, where τ is the lifetime of the Bessel flow of parameter $a = 2/\kappa$. Thus

$$\{\gamma \text{ hits } [x, \infty)\} = \{\tau(x/\sqrt{\kappa}) < \infty\}, \quad \{\gamma \text{ hits } [x, x+y)\} = \{\tau(x/\sqrt{\kappa}) < \tau((x+y)/\sqrt{\kappa})\}$$

and similarly

$$\{\gamma \text{ hits } [x, \infty) \text{ before } (-\infty, -y]\} = \{\tau(x/\sqrt{\kappa}) < \tau(-y/\sqrt{\kappa})\}.$$

Hence, from Proposition 10.1 we deduce:

(a) if $\kappa \in (0, 4]$ then $a \in [1/2, \infty)$, so $\mathbb{P}(\gamma \text{ hits } [x, \infty)) = 0$;

(b) if $\kappa \in (4, 8)$ then $a \in (1/4, 1/2)$, so

$$\mathbb{P}(\gamma \text{ hits } [x, \infty)) = 1, \quad \mathbb{P}(\gamma \text{ hits } [x, x+y)) = \phi\left(\frac{y}{x+y}\right)$$

and

$$\mathbb{P}(\gamma \text{ hits } [x, \infty) \text{ before } (-\infty, -y]) = \psi\left(\frac{y}{x+y}\right);$$

(c) if $\kappa \in [8, \infty)$ then $a \in (0, 1/4)$, so $\mathbb{P}(\gamma \text{ hits } [x, x+y)) = 1$.

Hence, in case (a),

$$\mathbb{P}(\gamma \text{ hits } \mathbb{R} \setminus \{0\}) = \lim_{n\to\infty} \mathbb{P}(\gamma \text{ hits } (-\infty, -1/n] \cup [1/n, \infty)) = 0$$

and, in case (c), we see that, almost surely, for all rationals $x, y \in (0, \infty)$, we have $\gamma_t \in [x, x+y)$ for some $t \geqslant 0$. Since γ is continuous, this implies that $[0, \infty) \subseteq \gamma[0, \infty)$ almost surely, and then $\mathbb{R} \subseteq \gamma[0, \infty)$ almost surely by symmetry. $\square$

11 Phases of SLE

Recall that one can scale a standard Brownian motion, either in time or space, to obtain a Brownian motion of any diffusivity. Thus "all Brownian motions look the same". In contrast, as the parameter κ is varied, SLE(κ) runs through three phases where it exhibits markedly different behaviour. The following results are proved in [4]. We will present proofs below for some of the easier cases.

Theorem 11.1. *Let $(\gamma_t)_{t\geqslant 0}$ be an SLE. Then $|\gamma_t| \to \infty$ as $t \to \infty$, almost surely.*

Theorem 11.2. *Let $(\gamma_t)_{t\geqslant 0}$ be an SLE(κ). Then*

(a) *for $\kappa \in [0,4]$, $(\gamma_t)_{t\geqslant 0}$ is a simple path almost surely;*

(b) *for $\kappa \in (4,8)$, $\cup_{t\geqslant 0} K_t = \mathbb{H}$ almost surely, but for each given $z \in \bar{\mathbb{H}} \setminus \{0\}$, $(\gamma_t)_{t\geqslant 0}$ does not hit z almost surely;*

(c) *for $\kappa \in [8,\infty)$, $\gamma[0,\infty) = \bar{\mathbb{H}}$ almost surely.*

The behaviour in case (b) is called *swallowing*, while in (c) we see that $(\gamma_t)_{t\geqslant 0}$ is a space-filling curve. We already saw in Proposition 10.3 that $\mathbb{R} \subseteq \gamma[0,\infty)$ almost surely when $\kappa \in [8,\infty)$ but will not prove the stronger statement (c) in these notes.

11.1 Simple phase

Proposition 11.3. *Let $(\gamma_t)_{t\geqslant 0}$ be an SLE(κ), with $\kappa \in (0,4]$. Then $(\gamma_t)_{t\geqslant 0}$ is a simple path almost surely.*

Proof. Recall the notation $K_{s,s+t} = g_s(K_{s+t} \setminus K_s)$ and $K_t^{(s)} = K_{s,s+t} - \xi_s$. By the domain Markov property, $(K_t^{(s)})_{t\geqslant 0}$ is an SLE(κ). By the Rohde–Schramm theorem, almost surely, for all rational $s \geqslant 0$ and all $t \geqslant 0$, $g_{K_{s,s+t}}^{-1}$ extends continuously to $\bar{\mathbb{H}}$ and $g_{K_{s,s+t}}^{-1}(z) \to \gamma_t^{(s)} + \xi_s$ as $z \to \xi_{s+t}$ with $z \in \mathbb{H}$. By Proposition 10.3, since $\kappa \leqslant 4$, almost surely, for all such s and t, $\gamma_t^{(s)} \in \{0\} \cup \mathbb{H}$. By Rohde–Schramm again

$$\gamma_{s+t} = \lim_{z\to\xi_{s+t}, z\in\mathbb{H}} g_s^{-1}(g_{K_{s,s+t}}^{-1}(z)) = g_s^{-1}(\gamma_t^{(s)} + \xi_s).$$

Since $\operatorname{hcap}(K_t) = 2t$ for all $t \geqslant 0$, almost surely, there is no non-degenerate interval on which $(\gamma_t)_{t\geqslant 0}$ is constant. Let $r, r' \geqslant 0$ with $r < r'$. Since $(\gamma_t)_{t\geqslant 0}$ is continuous, there exists a rational $s \in (r,r')$ such that $\gamma_s \neq \gamma_r$. Take $t = r' - s$. If $\gamma_t^{(s)} = 0$, then $\gamma_{r'} = \gamma_s$. If $\gamma_t^{(s)} \in \mathbb{H}$, then $\gamma_{r'} \in H_s \subseteq H_r$. In any case $\gamma_{r'} \neq \gamma_r$. $\square$

Lemma 11.4. *Let $(\gamma_t)_{t\geqslant 0}$ be a simple path in $\mathbb{H} \cup \{0\}$ starting from 0. Write $(\xi_t)_{t\geqslant 0}$ and $(g_t)_{t\geqslant 0}$ for the Loewner transform and flow associated to $(\gamma(0,t])_{t\geqslant 0}$, as usual. Fix $r \in (0,1)$, set $\tau = \inf\{t \geqslant 0 : |\gamma_t - 1| = r\}$ and suppose that $\tau < \infty$. Then*

$$|g_\tau(1) - \xi_\tau| \leqslant r.$$

Proof. Write $\gamma_\tau = a + ib$ and consider the line segments $I = (a, a + ib]$ and $J = [a \wedge 1, 1]$. Now g_τ extends continuously to $\mathbb{R}\backslash\{0\}$ and to γ_τ, with $g_\tau(\gamma_\tau) = \xi_\tau$. So the image $g_\tau(I \cup J)$ is a continuous path in $\bar{\mathbb{H}}$ joining ξ_τ and $[g_\tau(1), \infty)$. So, by conformal invariance of Brownian motion,

$$\begin{aligned}\mathbb{P}_{g_\tau(iy)}(B_{T(\mathbb{H})} \in [\xi_\tau, g_\tau(1)]) &\leqslant \mathbb{P}_{g_\tau(iy)}(B_{T(\mathbb{H}\backslash g_\tau(I))} \in g_\tau(I \cup J)) \\ &= \mathbb{P}_{iy}(B_{T(H_\tau \backslash I)} \in I \cup J) \leqslant \mathbb{P}_{iy}(\hat{B}_{T(\mathbb{H}\backslash I)} \in I^+ \cup J)\end{aligned}$$

where I^+ denotes the right side of I. Note that $g_I(a + ib) = a$ and $g_I(a+) = a + b$, and $g_I(1) = a + r$ when $a \leqslant 1$, so $\mathrm{Leb}(g_I(I^+ \cup J)) \leqslant r$. Recall that $g_\tau(iy) - iy \to 0$ as $y \to \infty$. Then, by Proposition 5.1, on multiplying by πy and letting $y \to \infty$, we obtain the desired estimate. $\square$

Proposition 11.5. *Let γ be an SLE(κ), with $\kappa \in (0, 4)$. Then $|\gamma_t| \to \infty$ as $t \to \infty$, almost surely.*

Proof. By Proposition 10.1, we know that $\inf_{t \geqslant 0}(g_t(1) - \xi_t) > 0$ almost surely. So, by the lemma, we must have, $\inf_{t \geqslant 0} |\gamma_t - 1| > 0$ almost surely. We know that g_1 extends continuously to $\mathbb{R} \setminus \{0\}$ and that $\gamma_1 \in \mathbb{H}$. Set $a^\pm = \lim_{x \downarrow 0} g_1(\pm x)$. Then $a^- < \xi_1 = g_1(\gamma_1) < a^+$. Set $r^\pm = \inf_{t \geqslant 0} |\gamma_t^{(1)} + \xi_1 - a^\pm|$ and set

$$N^\pm = \{z \in H_1 : |g_1(z) - a^\pm| < r^\pm\}, \quad N = N^- \cup \gamma(0, 1] \cup N^+.$$

Then $\gamma_t \notin N$ for all $t \geqslant 0$. By scaling and the Markov property, $r^\pm > 0$ almost surely. Since $[0, 1] \cup \gamma(0, 1]$ and $[-1, 0] \cup \gamma(0, 1]$ are simple paths, N is a neighbourhood of 0 in $\mathbb{H}$. Then $\liminf_{t \to \infty} |\gamma_t|$ is almost surely positive, and hence infinite, by scaling. $\square$

11.2 Swallowing phase

Proposition 11.6. *Let $(\gamma_t)_{t \geqslant 0}$ be an SLE(κ), with $\kappa \in (4, 8)$. Then $(\gamma_t)_{t \geqslant 0}$ is not a simple curve, nor a space-filling curve, almost surely.*

Proof. By Lemma 10.1, for any $x > 0$,

$$\mathbb{P}(\gamma \text{ hits } [x, \infty)) = 1,$$

and

$$\mathbb{P}(\gamma \text{ hits } [x, y]) = \mathbb{P}(\zeta(x) < \zeta(y)) = \phi\left(\frac{y - x}{y}\right) \in (0, 1).$$

Hence $\gamma_{\zeta(x)} \in (x, \infty)$ almost surely. Moreover, for $y > x$, we have $\{\gamma_{\zeta(x)} < y\} = \{\zeta(y) > \zeta(x)\}$ and $\{\gamma_{\zeta(x)} \geqslant y\} = \{\zeta(y) = \zeta(x)\}$ and both events have positive probability. In particular, we see that γ hits any given interval in $\mathbb{R}$ of positive length with positive probability. Now if S_1 is the set of all limit points of $g_1(\partial K_1 \cap \mathbb{H})$, then S_1 is an interval of positive length containing ξ_1. Thus we can find a subinterval $I \subset S_1$ such that $d(\xi_1, I) > 0$. Then by the above observation $g_1(\gamma(1, \infty)) \cap I$ is nonempty with positive probability. On the other hand, some topological considerations show that $\partial K_1 \cap \mathbb{H} \subseteq \gamma[0,1]$, so γ has double points with positive probability and hence almost surely by a zero-one argument (see below).

On the other hand, on $\{\gamma_{\zeta(x)} > y\}$, there is a neighbourhood of $[x,y]$ in $\mathbb{H}$ which does not meet γ and $\operatorname{dist}([x,y], H_{\zeta(x)}) > 0$. In particular, γ is not space-filling, with positive probability, and then almost surely.

Here is an elaboration of the zero-one argument for double points. Define, for $t > 0$, $A_t = \{\gamma_s = \gamma_{s'}$ for some distinct $s, s' \in [0,t]\}$. Then the sets A_t are non-decreasing in t and all have the same probability, p say, by scaling. But then $p = \mathbb{P}(\cap_t A_t)$ and $\cap_t A_t \in \mathcal{F}_{0+}$, where $\mathcal{F}_{0+} = \cap_{t>0} \sigma(\xi_s : s \leqslant t)$. But, by Blumenthal's zero-one law, $\mathcal{F}_{0+}$ contains only null sets and their complements. Hence $p \in \{0,1\}$. $\square$

Proposition 11.7. *Let $(\gamma_t)_{t\geqslant 0}$ be an SLE(κ), with $\kappa \in (4,8)$. Then* $\operatorname{dist}(0, H_t) \to \infty$*, in particular $|\gamma_t| \to \infty$, as $t \to \infty$, almost surely.*

Proof. The set S of limit points of $g_{\zeta(1)}(z)$ as $z \to 0$, $z \in \mathbb{H}$ is a compact (possibly empty) subset of $(-\infty, \xi_{\zeta(1)})$. Pick $y < \inf S$. With positive probability, $\operatorname{dist}(S, g_{\zeta(1)}(H_{\zeta(y)})) > 0$, so $\operatorname{dist}(0, H_{\zeta(y)}) > 0$, so $\mathbb{P}(\operatorname{dist}(0, H_t) > 0) = \delta$ for some $t > 0$ and $\delta > 0$. This extends to all t by scaling, with the same δ. So $\mathbb{P}(\operatorname{dist}(0, H_t) > 0$ for all $t > 0) = \delta$ and then $\delta = 1$ by a zero-one argument. Finally $\operatorname{dist}(0, H_t)$ is non-decreasing and, for all $r < \infty$, as $t \to \infty$,

$$\mathbb{P}(\operatorname{dist}(0, H_t) \leqslant r) = \mathbb{P}(\operatorname{dist}(0, H_1) \leqslant r/\sqrt{t}) \to 0.$$

$\square$

12 Conformal transformations of Loewner evolutions

A conformal isomorphism ϕ of initial domains in $\mathbb{H}$ takes one family of compact $\mathbb{H}$-hulls $(K_t)_{t\geqslant 0}$ to another $(\phi(K_t))_{t<T}$, defined up to the time T when $(K_t)_{t\geqslant 0}$ leaves the initial domain. We show that the local growth property is preserved under such a transformation, and obtain formulae for the half-plane capacity and Loewner transform of $(\phi(K_t))_{t<T}$.

12.1 Initial domains

By an *initial domain* (in $\mathbb{H}$) we mean a set $N\cup I$ where $N\subseteq\mathbb{H}$ is a simply connected domain and $I\subseteq\mathbb{R}$ is an open interval, such that N is a neighbourhood of I in $\mathbb{H}$. Thus $I\subseteq N^0$ in the notation of Section 4.2. An *isomorphism of initial domains* is a homeomorphism $\phi:N\cup I\to\tilde N\cup\tilde I$ which restricts to a conformal isomorphism $N\to\tilde N$. By Proposition 4.1, if $I\neq\mathbb{R}\neq\tilde I$, then, given points $x\in I$ and $\tilde x\in\tilde I$, there is a unique such isomorphism with $\phi(x)=\tilde x$, which then extends to a reflection-invariant conformal isomorphism $\phi^*:N_I^*\to\tilde N_{\tilde I}^*$. In this section, we suppose given an isomorphism of initial domains $\phi:N\cup I\to\tilde N\cup\tilde I$ and a compact $\mathbb{H}$-hull K with $\bar K\subseteq N\cup I$. Write $I=(x^-,x^+)$. Define

$$\tilde K=\phi(K),\quad \tilde H=\mathbb{H}\setminus\tilde K,\quad N_K=g_K(N\setminus K),\quad I_K=(g_K^*(x^-),g_K^*(x^+)).$$

Note that $\tilde H$ is not the image of $H=\mathbb{H}\setminus K$ under ϕ, nor is I_K the image of I under g_K^*. Nevertheless, we now show that $\tilde H$ is simply connected and N_K is a neighbourhood of I_K in $\mathbb{H}$. You are advised to sketch an example as you follow the results in this section. The proofs could be skipped in a first reading.

Proposition 12.1. *The set $\tilde K$ is a compact $\mathbb{H}$-hull with $\bar{\tilde K}\subseteq\tilde N\cup\tilde I$ and the set $N_K\cup I_K$ is an initial domain.*

Proof ($\star$). Since ϕ^* is a homeomorphism and $\bar K\subseteq N\cup I$, we have $\bar{\tilde K}=\phi^*(\bar K)\subseteq\tilde N\cup\tilde I$. Since $\bar K$ is compact, this also shows that $\tilde K$ is bounded.

Pick $x\in I$ and consider the conformal isomorphism $\psi:\mathbb{D}\to N_I^*$ such that $\psi(0)=x$ and $\psi'(0)>0$. Fix $r\in(0,1)$ and for $\theta\in[0,\pi]$ define $p(\theta)=\psi(re^{i\theta})$. Then $p=(p(\theta):\theta\in(0,\pi))$ is a simple curve in N and $p(0),p(\pi)\in I$. We can and do choose r so that $p(\theta)\in H^*$ for all $\theta\in[0,\pi]$. Then $\phi(p)$ and $g_K(p)$ are simple curves in $\mathbb{H}$ which each disconnect $\mathbb{H}$ in two components. Write D_0 for the bounded component of $\mathbb{H}\setminus g_K(p)$ and D_1 for the unbounded component of $\mathbb{H}\setminus\phi(p)$. Then $D_1\cup\phi(p)$ is simply connected and $D_1\subseteq\tilde H$. On the other hand $D_0\cup g_K(p)$ is also simply connected and $\phi\circ g_K^{-1}$ is a homeomorphism $D_0\cup g_K(p)\to\tilde H\setminus D_1$. Hence $\tilde H=\phi(g_K^{-1}(D_0))\cup\phi(p)\cup D_1$ is simply connected.

Finally, given $y^-,y^+\in I\setminus\bar K$ with $y^-<y^+$ we can choose r so that $p(0)>y^+$ and $p(\pi)<y^-$. Then D_0 is a neighbourhood of $(g_K^*(y^-),g_K^*(y^+))$ in $\mathbb{H}$. But $D_0\subseteq N_K$. Hence N_K is a neighbourhood of I_K in $\mathbb{H}$. $\square$

Define $\tilde N_{\tilde K}$ and $\tilde I_{\tilde K}$ analogously to N_K and I_K and define $\phi_K:N_K\to\tilde N_{\tilde K}$ by

$$\phi_K=g_{\tilde K}\circ\phi\circ g_K^{-1}.$$

Proposition 12.2. *The map ϕ_K extends to an isomorphism $N_K \cup I_K \to \tilde{N}_{\tilde{K}} \cup \tilde{I}_{\tilde{K}}$ of initial domains.*

Proof ($\star$). Write I^- and I^+ for the leftmost and rightmost component intervals of the open set $I \setminus \bar{K} \subseteq \mathbb{R}$. Set $J^\pm = g_K^*(I^\pm)$ and $J = J^- \cup J^+$. Define similarly $\tilde{J}^\pm$ and $\tilde{J}$ starting from $\tilde{I}$ and $\tilde{K}$. Then $J \subseteq I_K$ and $I_K \setminus J$ is a compact subset of I_K. A similar statement holds for $\tilde{J}$ and $\tilde{I}_{\tilde{K}}$. Define $\psi : (N_K)^*_J \to (\tilde{N}_{\tilde{K}})^*_{\tilde{J}}$ by $\psi = g^*_{\tilde{K}} \circ \phi^* \circ (g_K^*)^{-1}$. Then ψ is a holomorphic extension of ϕ_K which takes J^- to $\tilde{J}^-$ and J^+ to $\tilde{J}^+$. Since N_K is a neighbourhood of I_K in $\mathbb{H}$, we have $I_K \subseteq \hat{N}_K$ by Proposition 4.1, and similarly $\tilde{I}_{\tilde{K}} \subseteq \hat{\tilde{N}}_{\tilde{K}}$. Write $\hat{\phi}_K$ for the extension of ϕ_K as a homeomorphism $\hat{N}_K \to \hat{\tilde{N}}_{\tilde{K}}$. Then $\hat{\phi}_K = \psi$ on J, so we must have $\hat{\phi}_K(I_K) = \tilde{I}_{\tilde{K}}$, and so ϕ extends to a homeomorphism $N_K \cup I_K \to \tilde{N}_{\tilde{K}} \cup \tilde{I}_{\tilde{K}}$ as required. □

Recall from Proposition 6.3 the scaling property $\operatorname{hcap}(rK) = r^2 \operatorname{hcap}(K)$. This makes it plausible, for a conformal isomorphism ϕ of some initial domain $N \cup I$ and for a small hull K near $\xi \in I$, that $\phi'(\xi)^2 \operatorname{hcap}(K)$ is a good approximation for $\operatorname{hcap}(\phi(K))$. We now prove such an estimate, in a normalized form.

Proposition 12.3. *There is an absolute constant $C < \infty$ with the following property. Let $\phi : N \cup I \to \tilde{N} \cup \tilde{I}$ be an isomorphism of initial domains. Assume that $0 \in I$ and $\phi(0) = 0$ and $\phi'(0) = 1$. Let $K \subseteq N$ be a compact $\mathbb{H}$-hull. Suppose that for some $0 < r < \varepsilon < R < \infty$ we have*

$$K \cup \phi(K) \subseteq r\mathbb{D}, \quad (\varepsilon\mathbb{D}) \cap \mathbb{H} \subseteq N \cup \tilde{N} \subseteq R\mathbb{D}.$$

Then

$$1 - CrR/\varepsilon^2 \leqslant \frac{\operatorname{hcap}(\phi(K))}{\operatorname{hcap}(K)} \leqslant 1 + CrR/\varepsilon^2.$$

Proof ($\star$). It will suffice to prove the upper bound. The lower bound then follows by interchanging the roles of $N \cup I$ and $\tilde{N} \cup \tilde{I}$. Recall the formula (14), valid for $K \subseteq \mathbb{D}$,

$$\operatorname{hcap}(K) = \int_0^\pi \mathbb{E}_{e^{i\theta}}(\operatorname{Im} B_{T(H)}) \frac{2\sin\theta}{\pi} d\theta.$$

Fix $\alpha \geqslant 1$. Since $K \subseteq r\mathbb{D}$, we can apply this to $\sigma^{-1}K$ for $\sigma \in [r, \alpha r]$ and use the scale invariance of Brownian motion to obtain

$$\sigma \operatorname{hcap}(K) = \int_0^\pi \mathbb{E}_{\sigma e^{i\theta}}(\operatorname{Im} B_{T(H)}) \frac{2\sigma \sin\theta}{\pi} \sigma d\theta.$$

Next, integrate over σ to obtain

$$\frac{(\alpha^2 - 1)r^2}{2} \operatorname{hcap}(K) = \int_{S(r,\alpha r)} \mathbb{E}_z(\operatorname{Im} B_{T(H)}) \frac{2 \operatorname{Im} z}{\pi} A(dz), \tag{29}$$

where $A(dz)$ denotes area measure and $S(r, \alpha r)$ is the half-annulus $\{z \in \mathbb{H} : r \leqslant |z| \leqslant \alpha r\}$. Set $\psi = \phi^{-1}$. By conformal invariance of Brownian motion,

$$\mathbb{E}_w(\operatorname{Im} B_{T(\tilde{H})}) = \mathbb{E}_{\psi(w)}(\operatorname{Im} \phi(B_{T(H)})).$$

Apply the identity (29) to $\phi(K)$, replacing r by $\rho \geqslant r$ and taking $\alpha = 2$ to obtain

$$
\begin{aligned}
\frac{3\rho^2}{2}\operatorname{hcap}(\phi(K)) &= \int_{S(\rho,2\rho)} \mathbb{E}_{\psi(w)}(\operatorname{Im}\phi(B_{T(H)}))\frac{2\operatorname{Im} w}{\pi}A(dw) \\
&= \int_{\psi(S(\rho,2\rho))} \mathbb{E}_{z}(\operatorname{Im}\phi(B_{T(H)}))\frac{2\operatorname{Im}\phi(z)}{\pi}|\phi'(z)|^2 A(dz) \qquad (30)
\end{aligned}
$$

where we made the change of variable $z = \psi(w)$ for the second equality.

We apply Cauchy's integral formula to ϕ^* and ψ^* to see that, for $|z| \leqslant 1/2$, we have $|\phi''(z)| \leqslant 8R$ and $|\psi''(z)| \leqslant 8R$. Then, by Taylor's theorem, using $\phi(0) = \psi(0) = 0$, $\phi'(0) = \psi'(0) = 1$ and the fact that ϕ is real on I, we obtain for $|z| \leqslant 1/2$

$$
|\phi'(z)| \leqslant 1 + 8R|z|, \quad \operatorname{Im}\phi(z) \leqslant (1 + 16|z|R)\operatorname{Im} z, \quad |\psi(z) - z| \leqslant 4R|z|^2.
$$

Assume that $48rR \leqslant 1$ and take $\alpha = 2(1+48rR)$ then $\alpha \leqslant 4$. Note that $r \leqslant 2r - 4R(2r)^2$. Set $\rho = \inf\{s \geqslant r : r = s - 4Rs^2\}$. Then $\rho \leqslant 2r \leqslant 1/4$. Hence, for $z \in S(\rho, 2\rho)$, we have $|\psi(z)| \geqslant \rho - 4R\rho^2 = r$ and $|\psi(z)| \leqslant 2\rho + 16R\rho^2 = 2r + 24R\rho^2 \leqslant \alpha r \leqslant 4r \leqslant 1/2$ so $\psi(S(\rho, 2\rho)) \subseteq S(r, \alpha r)$. A comparison of (29) and (30) then yields

$$
\operatorname{hcap}(\phi(K)) \leqslant (1 + 16rR)(1 + 64rR)(1 + 32rR)^2(1 + 192rR)\operatorname{hcap}(K)
$$

which in turns yields the claimed estimate for a suitable choice of the constant C. $\square$

More generally, for any isomorphism of initial domains $\phi : N \cup I \to \tilde{N} \cup \tilde{I}$, any $\xi \in I$, and any compact $\mathbb{H}$-hull $K \subseteq N$, the preceding estimate can be applied to the map $\bar{\phi}(z) = \phi'(\xi)^{-1}(\phi(z + \xi) - \phi(\xi))$ to obtain the estimate

$$
(1 - \bar{C}rR/\varepsilon^2)\phi'(\xi)^2\operatorname{hcap}(K) \leqslant \operatorname{hcap}(\phi(K)) \leqslant (1 + \bar{C}rR/\varepsilon^2)\phi'(\xi)^2\operatorname{hcap}(K) \qquad (31)
$$

where $\bar{C} = C\max\{\phi'(\xi)^2, \phi'(\xi)^{-2}\}$, whenever $K \subseteq \xi + r\mathbb{D}$ and $\phi(K) \subseteq \phi(\xi) + r\mathbb{D}$ and

$$
\xi + (\varepsilon\mathbb{D}) \cap \bar{\mathbb{H}} \subseteq N \cup I \subseteq \xi + R\mathbb{D}, \quad \phi(\xi) + (\varepsilon\mathbb{D}) \cap \bar{\mathbb{H}} \subseteq \tilde{N} \cup \tilde{I} \subseteq \phi(\xi) + R\mathbb{D}.
$$

The details are left as an exercise.

12.2 Loewner evolution and isomorphisms of initial domains

Let $(K_t)_{t\geqslant 0}$ be an increasing family of compact $\mathbb{H}$-hulls with the local growth property. Write $(\xi_t)_{t\geqslant 0}$ for the Loewner transform of $(K_t)_{t\geqslant 0}$. Let $N \cup I$ and $\tilde{N} \cup \tilde{I}$ be initial domains, with $\xi_0 \in I$ and let $\phi : N \cup I \to \tilde{N} \cup \tilde{I}$ be an isomorphism. Set $T = \inf\{t \geqslant 0 : \bar{K}_t \not\subseteq N \cup I\}$. For $t < T$, we consider the compact $\mathbb{H}$-hull $\tilde{K}_t = \phi(K_t)$ and other associated objects, as in the preceding section, writing now

$$
g_t = g_{K_t}, \quad \tilde{g}_t = g_{\tilde{K}_t}, \quad \phi_t = \phi_{K_t} = \tilde{g}_t \circ \phi \circ g_t^{-1}, \quad \tilde{\xi}_t = \phi_t(\xi_t)
$$

and

$$
N_t = N_{K_t}, \quad I_t = I_{K_t}, \quad \tilde{N}_t = \tilde{N}_{\tilde{K}_t}, \quad \tilde{I}_t = \tilde{I}_{\tilde{K}_t}.
$$

Proposition 12.4. *The increasing family of compact $\mathbb{H}$-hulls $(\tilde{K}_t)_{t<T}$ has the local growth property and has Loewner transform $(\tilde{\xi}_t)_{t<T}$.*

Proof ($\star$). Fix $t_0 \in [0,T)$. Let ψ be as in the proof of Proposition 12.1 and choose $r \in (0,1)$ so that $K_{t_0} \subseteq \psi(r\mathbb{D})$. It will suffice to prove the proposition with $N \cup I$ replaced by $\psi(r\mathbb{D} \cap \bar{\mathbb{H}})$, which is the bounded component of $\bar{\mathbb{H}} \setminus \{\psi(re^{i\theta}) : \theta \in [0,\pi]\}$, and with ϕ replaced by its restriction to $\psi(r\mathbb{D} \cap \bar{\mathbb{H}})$. Hence we may assume without loss that $N \cup I$ is the the bounded component of $\bar{\mathbb{H}} \setminus p$, for some simple curve $p = (p(\theta) : \theta \in [0,\pi])$ with $p(0), p(\pi) \in \mathbb{R}$ and $p(\theta) \in \mathbb{H}$ for all $\theta \in (0,\pi)$, and that ϕ extends to a homeomorphism $\bar{N} \to \bar{\tilde{N}}$.

For $t \leqslant t_0$ and $z, z' \in N \setminus K_t$, we have

$$|g_t(z) - g_t(z')| \leqslant |g_t(z) - z| + |z - z'| + |z' - g_t(z')| \leqslant 6\,\mathrm{rad}(K_t) + 2\,\mathrm{rad}(N) \leqslant 8\,\mathrm{rad}(N).$$

Hence, using a similar estimate for $\tilde{N}$ and reflection symmetry, we have[9]

$$N_t^* \subseteq \xi_t + R\mathbb{D}, \quad \tilde{N}_t^* \subseteq \tilde{\xi}_t + R\mathbb{D}. \tag{32}$$

where $R = 8\max\{\mathrm{rad}(N), \mathrm{rad}(\tilde{N})\} < \infty$. The maps

$$(t,\theta) \mapsto |g_t^*(p(\theta)) - \xi_t|, \quad (t,\theta) \mapsto |\tilde{g}_t^*(\phi(p(\theta))) - \tilde{\xi}_t|$$

are continuous and positive on $[0,t_0] \times [0,\pi]$, hence are bounded below, by $\varepsilon > 0$ say. Then, for all $t \leqslant t_0$, we have

$$\xi_t + \varepsilon\mathbb{D} \subseteq N_t^*, \quad \tilde{\xi}_t + \varepsilon\mathbb{D} \subseteq \tilde{N}_t^*. \tag{33}$$

Since $\phi_t^* : N_t^* \to \tilde{N}_t^*$ is a conformal isomorphism, it follows by Cauchy's integral formula that

$$|\phi_t'(z)| \leqslant 2R/\varepsilon, \quad z \in \xi_t + (\varepsilon/2)\mathbb{D} \cap \mathbb{H}. \tag{34}$$

Now, for all $r \in (0, \varepsilon/2]$, we can find $h > 0$ such that, for all $t \leqslant t_0$, we have

$$K_{t,t+h} \subseteq \xi_t + r\mathbb{D} \tag{35}$$

and then, setting $\rho = 2R/\varepsilon$,

$$\tilde{K}_{t,t+h} = \phi_t(K_{t,t+h}) \subseteq \tilde{\xi}_t + \rho r\mathbb{D}. \tag{36}$$

Hence $(\tilde{K}_t)_{t \leqslant t_0}$ has the local growth property and has Loewner transform $(\tilde{\xi}_t)_{t \leqslant t_0}$. □

Proposition 12.5. *For all $t \in [0,T)$, we have*[10]

$$\mathrm{hcap}(\tilde{K}_t) = \int_0^t \phi_s'(\xi_s)^2 d(\mathrm{hcap}(K_s)). \tag{37}$$

[9] Only one of the two inclusions in (32) and one of those in (33) are used in this proof. We shall need all of them for the proof of Proposition 12.5.

[10] A shorter proof of this formula is possible using Proposition 6.5 to compare $\mathrm{hcap}(\tilde{K}_{t,t+h})$ and $\mathrm{hcap}(\phi_t'(\xi_t)K_{t,t+h})$, provided $\mathrm{rad}(K_{t,t+h})^{5/2}/\,\mathrm{hcap}(K_{t,t+h}) \to 0$ as $h \to 0$ uniformly on compacts in t. The estimate (21) shows this condition holds provided $(\xi_t)_{t\geqslant 0}$ is Hölder of exponent greater than $2/5$, so this covers the case of SLE. We have given the longer argument to avoid any spurious condition and because it is also more elementary, in that it does not rely on Beurling's estimate, used for Proposition 6.5.

Proof ($\star$). Fix $t_0 \in [0,T)$ and follow the same reduction as in Proposition 12.4, introducing constants R, ε and $\rho = 2R/\varepsilon$. For $t \leqslant t_0$, from (34), we see that $|\phi_t'(\xi_t)| \leqslant \rho$. On the other hand, by considering the inverse map $\psi_t^* : \tilde{N}_t^* \to N_t^*$, we obtain similarly $|\phi_t'(\xi_t)| = |\psi_t'(\tilde{\xi}_t)|^{-1} \geqslant 1/\rho$. Given $\delta \in (0,1]$, choose $r > 0$ so that $CrR\rho^3 \leqslant \varepsilon^2\delta$. There exists an $h > 0$ such that, for all $t \leqslant t_0$,

$$K_{t,t+h} \subseteq \xi_t + r\mathbb{D}.$$

Then, using the estimates (32), (33), (35) and (36), for $s \in (0,h)$, we can apply the estimate (31) to the isomorphism $\phi_t : N_t \cup I_t \to \tilde{N}_t \cup \tilde{I}_t$ and the compact $\mathbb{H}$-hull $K_{t,t+s}$ to obtain

$$(1-\delta)\,\phi_t'(\xi_t)^2\,\mathrm{hcap}(K_{t,t+s}) \leqslant \mathrm{hcap}(\tilde{K}_{t,t+s}) \leqslant (1+\delta)\,\phi_t'(\xi_t)^2\,\mathrm{hcap}(K_{t,t+s}).$$

Now, for all $n \in \mathbb{N}$, setting $s = t_0/n$, we have

$$\mathrm{hcap}(\tilde{K}_{t_0}) = \sum_{j=0}^{n-1} \mathrm{hcap}(\tilde{K}_{js,(j+1)s}).$$

For $n > t_0/h$, we can apply the bounds just obtained with $t = js$ and sum over j to obtain

$$(1-\delta)\sum_{j=0}^{n-1} \phi_{js}'(\xi_{js})^2\,\mathrm{hcap}(K_{js,(j+1)s}) \leqslant \mathrm{hcap}(\tilde{K}_{t_0}) \leqslant (1+\delta)\sum_{j=0}^{n-1} \phi_{js}'(\xi_{js})^2\,\mathrm{hcap}(K_{js,(j+1)s}).$$

Let $n \to \infty$ and then $\delta \to 0$ to obtain the claimed identity. $\square$

Proposition 12.6. *The set $S = \{(t,z) : t \in [0,T), z \in N_t \cup I_t\}$ is open in $[0,\infty) \times \bar{\mathbb{H}}$. The function $(t,z) \mapsto \phi_t(z)$ on S is differentiable in t for all z, with derivative given by*

$$\dot{\phi}_t(z) = \frac{2\phi_t'(\xi_t)^2}{\phi_t(z) - \phi_t(\xi_t)} - \phi_t'(z)\frac{2}{z-\xi_t}, \quad z \in N_t \cup I_t \setminus \{\xi_t\} \tag{38}$$

and

$$\dot{\phi}_t(\xi_t) = -3\phi_t''(\xi_t). \tag{39}$$

Moreover, $\dot{\phi}_t$ is holomorphic on $N_t \cup I_t$, with derivative given by

$$\dot{\phi}_t'(z) = 2\left(-\frac{\phi_t'(\xi_t)^2\phi_t'(z)}{(\phi_t(z) - \phi_t(\xi_t))^2} + \frac{\phi_t'(z)}{(z-\xi_t)^2} - \frac{\phi_t''(z)}{z-\xi_t}\right), \quad z \in N_t \cup I_t \setminus \{\xi_t\} \tag{40}$$

and

$$\dot{\phi}_t'(\xi_t) = \frac{1}{2}\frac{\phi_t''(\xi_t)^2}{\phi_t'(\xi_t)} - \frac{4}{3}\phi_t'''(\xi_t). \tag{41}$$

Proof. By Propositions 8.5 and 12.2, when reparametrized by hcap, $(\tilde{g}_t)_{t<T}$ satisfies Loewner's equation driven by $(\tilde{\xi}_t)_{t<T}$. So, by Proposition 12.5, we obtain

$$\dot{\tilde{g}}_t(z) = 2\phi_t'(\xi_t)^2/(\tilde{g}_t(z) - \tilde{\xi}_t), \quad z \in \tilde{H}_t.$$

Set $f_t = g_t^{-1}$ and differentiate the equation $f_t(g_t(z)) = z$ in t to obtain

$$\dot{f}_t(z) = -2f_t'(z)/(z - \xi_t), \quad z \in \mathbb{H}.$$

For $z \in N_t$ we have $\phi_t(z) = \tilde{g}_t(\phi(f_t(z)))$. By the chain rule, for $t \in [0, T)$ and $z \in N_t$, we see that $\phi_t(z)$ is differentiable in t, with derivative given by (38), which is then holomorphic in z with derivative given by (40). Note that the functions on the right hand sides of (38) and (40) are continuous in $z \in N_t \cup I_t \setminus \{\xi_t\}$. It is straightforward to check using l'Hôpital's rule that they extend continuously to ξ_t with the values given in (39) and (41). Then for $x \in I_t$ and $z \in N_t$, the functions $\phi_s(z)$ and $\dot{\phi}_s(z)$ and $\dot{\phi}_s'(z)$ converge as $z \to x$ locally uniformly for s near t. The result follows by standard arguments. □

13 SLE(6), locality and percolation

13.1 Locality

SLE(6) has a special invariance property called *locality* which can be understood informally as meaning that, in its general formulation as a measure on chords in (D, z_0, z_1), it does not know what domain it is in beyond the fact that, each time it hits the boundary δD, it turns towards its endpoint z_1, as it must do in order to satisfy the non-crossing property. By Smirnov's theorem SLE(6) is the scaling limit of critical site percolation on the planar hexagonal lattice. Thus, if the upper half-plane is tiled with yellow and blue hexagons, with the colours at each site independent and equally likely, and if we place blue hexagons along the positive real axis and yellow ones along the negative real axis, then the unique blue/yellow interface joining 0 and ∞ converges weakly to an SLE(6) in the limit of small lattice spacing. The lattice model has its own obvious locality property, so the fact that locality implies $\kappa = 6$ for SLE was an early clue towards Smirnov's result.

Theorem 13.1. *Let $\phi : N \cup I \to \tilde{N} \cup \tilde{I}$ be an isomorphism of initial domains with $0 \in I$ and $0 = \phi(0) \in \tilde{I}$. Let $(\gamma_t)_{t \geqslant 0}$ be an SLE(6). Set*

$$T = \inf\{t \geqslant 0 : \gamma_t \notin N \cup I\}, \quad \tilde{T} = \inf\{t \geqslant 0 : \gamma_t \notin \tilde{N} \cup \tilde{I}\}.$$

Then $(\phi(\gamma_t))_{t<T}$ in its canonical reparametrization has the same distribution as $(\gamma_t)_{t<\tilde{T}}$.

Proof. Write $(K_t)_{t \geqslant 0}$ for the family of compact $\mathbb{H}$-hulls generated by $(\gamma_t)_{t \geqslant 0}$ and write $(\xi_t)_{t \geqslant 0}$ for its Loewner transform, which is a Brownian motion of diffusivity 6. For $t < T$, set $\tilde{K}_t = \phi(K_t)$ and $\phi_t = g_{\tilde{K}_t} \circ \phi \circ (g_{K_t})^{-1}$. By Propositions 12.2 and 12.5, $(\tilde{K}_t)_{t<T}$ is a family of compact $\mathbb{H}$-hulls having the local growth property, whose Loewner transform $(\tilde{\xi}_t)_{t<T}$ and half-plane capacity are given by

$$\tilde{\xi}_t = \phi_t(\xi_t), \quad \operatorname{hcap}(\tilde{K}_t) = 2\int_0^t \phi_s'(\xi_s)^2 ds.$$

The set $S_0 = \{(t,x) : t \in [0,T), x \in I_{K_t}\}$ is open in $[0,\infty) \times \mathbb{R}$ and $\xi_t \in I_{K_t}$ for all $t < T$. By Proposition 12.6, the adapted random map $(t,x) \mapsto \phi_t(x) : S_0 \to \mathbb{R}$ is $C^{1,2}$ with $\dot{\phi}_t(\xi_t) = -3\phi_t''(\xi_t)$ for all $t < T$. By the generalized Itô formula, we have

$$d\tilde{\xi}_t = \dot{\phi}_t(\xi_t)dt + \phi_t'(\xi_t)d\xi_t + \frac{1}{2}\phi_t''(\xi_t)d\xi_t d\xi_t.$$

Since $d\xi_t d\xi_t = 6dt$, the finite variation terms cancel and we see that $(\tilde{\xi}_t)_{t<T}$ is a continuous local martingale with quadratic variation $[\tilde{\xi}]_t = 3\operatorname{hcap}(\tilde{K}_t)$. The canonical reparametrization $(\tilde{K}_{\tau(s)})_{s<S}$ of $(\tilde{K}_t)_{t<T}$ and its Loewner transform $(\eta_s)_{s<S}$ are given by

$$\operatorname{hcap}(\tilde{K}_{\tau(s)}) = 2s, \quad \operatorname{hcap}(\tilde{K}_T) = 2S, \quad \eta_s = \tilde{\xi}_{\tau(s)}.$$

Now (by optional stopping) $(\eta_s)_{s<S}$ is a continuous local martingale (in its own filtration) and its quadratic variation is given by $[\eta]_s = [\tilde{\xi}]_{\tau(s)} = 6s$. Hence, by Lévy's characterization,

$(\eta_s)_{s<S}$ extends[11] to a Brownian motion $(\eta_s)_{s\geqslant 0}$ of diffusivity 6. Write $\tilde{\gamma}$ for the SLE(6) driven by $(\eta_s)_{s\geqslant 0}$, then $\phi(\gamma_{\tau(s)}) = \tilde{\gamma}_s$ for $s < S$ and $S = \inf\{s \geqslant 0 : \tilde{\gamma}_s \notin \tilde{N} \cup \tilde{I}\}$. Hence $(\phi(\gamma_{\tau(s)}))_{s<S}$ and $(\gamma_s)_{s<\tilde{T}}$ have the same distribution as required. □

By an *initial domain* $N \cup I$ in a two-pointed domain (D, z_0, z_1) we mean a simply connected subdomain $N \subseteq D$ along with an interval I of $\delta D \setminus \{z_1\}$ containing z_0 such that N is a neighbourhood of I in D. Note that, if we choose a conformal isomorphism ϕ from (D, z_0, z_1) to $(\mathbb{H}, 0, \infty)$, then $\phi(N) \cup \phi(I)$ is an initial domain in $(\mathbb{H}, 0, \infty)$, which is just an initial domain in $\mathbb{H}$ in the sense of Section 12.1 such that $0 \in I_0$. We can now give a precise version of the informal account of locality which began this section.

Theorem 13.2. *Let γ be an SLE(6) in (D, z_0, z_1) and let $\tilde{\gamma}$ be an SLE(6) in $(\tilde{D}, z_0, \tilde{z}_1)$. Suppose that (D, z_0, z_1) and $(\tilde{D}, z_0, \tilde{z}_1)$ share an initial domain $N_0 \cup I_0$. Then the stopping times*

$$T = \inf\{t \geqslant 0 : \gamma_t \notin N_0 \cup I_0\}, \quad \tilde{T} = \inf\{t \geqslant 0 : \tilde{\gamma}_t \notin N_0 \cup I_0\}.$$

are parametrization-invariant and the chords $(\gamma_t)_{t<T}$ and $(\tilde{\gamma}_t)_{t<\tilde{T}}$ have the same distribution.

Proof. Choose conformal isomorphisms ϕ of (D, z_0, z_1) to $(\mathbb{H}, 0, \infty)$ and $\tilde{\phi}$ of $(\tilde{D}, z_0, \tilde{z}_1)$ to $(\mathbb{H}, 0, \infty)$. Consider the initial domains $N \cup I = \phi(N_0) \cup \phi(I_0)$ and $\tilde{N} \cup \tilde{I} = \tilde{\phi}(N_0) \cup \tilde{\phi}(I_0)$. Then $\psi = \tilde{\phi} \circ \phi^{-1}$ gives an isomorphism $N \cup I \to \tilde{N} \cup \tilde{I}$. By conformal invariance, $\phi(\gamma)$ and $\tilde{\phi}(\tilde{\gamma})$ are both SLE(6) in $(\mathbb{H}, 0, \infty)$. So the claimed identity in distribution follows from Theorem 13.1. □

13.2 SLE(6) in a triangle

While physicists investigated critical percolation using nonrigorous methods, Cardy established a formula for the limiting crossing probabilities of a rectan gle. Carleson observed that this formula became considerably simpler on a triangle. The corresponding formula can be stated as a theorem directly for SLE(6). In turn, since Smirnov proved that Cardy's formula holds in the limit for critical percolation, this provides another of identifying SLE(6) as the uniqu e possible limit for the scaling limit of cluster interface exploration process in critical percolation.

Let Δ be the equilateral triangle with vertices $a = 0$, $b = 1$, $c = e^{i\pi/3}$.

Theorem 13.3. *Let γ be SLE(6) in $(\Delta, 0, 1)$, where Δ denotes the triangle with vertices $0, 1, e^{\pi i/3}$. Then the point X at which γ hits the edge $[1, e^{\pi i/3}]$ is uniformly distributed.*

Proof. The Schwarz–Christoffel transformation $(\mathbb{H}, 0, 1, \infty) \to (\Delta, 0, 1, e^{\pi i/3})$ is given by

$$f(z) = c \int_0^z \frac{dw}{w^{2/3}(1-w)^{2/3}}, \quad c = \frac{\Gamma(2/3)}{\Gamma(1/3)^2}.$$

[11] We know that $T < \infty$ almost surely, so we can do this here without extending the probability space, using $(\xi_{T+t} - \xi_T)_{t\geqslant 0}$.

Consider the map $z \mapsto \varphi(z) = 1/(1-z)$. This is a conformal automorphism which cyclically permutes $0, 1, \infty$. The map $z \mapsto g(z) = 1+e^{2i\pi/3}z$ is a conformal automorphism of Δ which cyclically permutes a, b, c. Thus

$$f(\varphi(z)) = g(f(z)),$$

by uniqueness of the Riemann map. Thus, composing by $\varphi^{-1}(z) = (z-1)/z$, we deduce

$$f(z) = 1 + e^{2i\pi/3} f((z-1)/z)$$

for all $z \in \mathbb{H}$. This identity extends by continuity when $z \to x \in \mathbb{H}$.

Let $x \in [0,1]$ and choose y so that $f(y/(1+y)) = x$. Then, by conformal invariance and Proposition 10.3,

$$\mathbb{P}(X \in [1, 1+xe^{2i\pi/3}]) = \mathbb{P}(\text{SLE}(6) \text{ in } (\mathbb{H}, 0, \infty) \text{ hits } [1, 1+y]) = \phi\left(\frac{y}{1+y}\right) = x.$$

Thus X is uniform on $[1, e^{i\pi/3}]$. $\square$

13.3 Smirnov's theorem

We now discuss Smirnov's proof of Cardy's formula for percolation on the triangular lattice. Consider the lattice of edge length δ. Sites of the lattice are coloured black or white independently with probability 1/2. Take any Jordan domain D with three distinct boundary points $a(1), a(\tau), a(\tau^2)$, ordered positively, where $\tau = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Write Φ for the unique conformal isomorphism from D to the triangle Δ with corresponding boundary points $1, \tau, \tau^2$. For $z \in D$ and $\alpha \in \{1, \tau, \tau^2\}$, write $Q_\alpha(z)$ for the event that z is separated from the boundary segment $a(\tau\alpha)a(\tau^2\alpha)$ by a simple black path from $a(\alpha)a(\tau\alpha)$ to $a(\tau^2\alpha)a(\alpha)$. Set $H_\alpha(z) = H_\alpha^\delta(z) = \mathbb{P}(Q_\alpha(z))$. By a black path we mean any path in the lattice which visits only black points. The functions $H_\alpha(z)$ are constant in the interior of lattice triangles with discontinuities at the edges. Let f_α denote the unique affine function on Δ with $f_\alpha(\alpha) = 1$ and $f_\alpha(\tau\alpha) = f_\alpha(\tau^2\alpha) = 0$, and set $h_\alpha = f_\alpha \circ \Phi$.

Theorem 13.4 (Smirnov)**.** *For $\alpha = 1, \tau, \tau^2$, H_α^δ converges uniformly on D to h_α as $\delta \to 0$.*

It follows, in particular, by taking $z \in \partial D$, that the asymptotic crossing probabilities for this percolation model are indeed conformally invariant and are given by Cardy's formula.

Before sketching the proof, we will describe a variant of the Cauchy–Riemann equations and of conjugate harmonic functions, associated with the angle $2\pi/3$. For $\alpha = 1, \tau, \tau^2$, and f analytic, set

$$f_\alpha = \text{Re}(f/\alpha).$$

Then f_α is harmonic and we can recover f by

$$\alpha f = f_\alpha + \frac{i}{\sqrt{3}}(f_{\alpha\tau} - f_{\alpha\tau^2}).$$

Also, for any $\eta \in \mathbb{C}$, the directional derivatives satisfy

$$\nabla_\eta f_\alpha(z) = \frac{\partial}{\partial \varepsilon} |_{\varepsilon=0} \operatorname{Re}\left(\frac{f(z+\varepsilon\eta)}{\alpha}\right) = Re\left(\frac{f'(z)\eta}{\alpha}\right) = \nabla_{\tau\eta} f_{\tau\alpha}(z).$$

These are the $2\pi/3$-*Cauchy–Riemann equations*, and $(f_1, f_\tau, f_{\tau^2})$ is the *harmonic triple* of f.

Conversely, if we are given C^1 functions f_1, f_τ, f_{τ^2} such that, for $\alpha \in \{1, \tau, \tau^2\}$, for all η,

$$\nabla_\eta f_\alpha(z) = \nabla_{\tau\eta} f_{\tau\alpha}(z),$$

then f, defined by

$$f = f_1 + \frac{i}{\sqrt{3}}(f_\tau - f_{\tau^2}),$$

is holomorphic and $f_\alpha = \operatorname{Re}(f/\alpha)$ for all α.

Sketch proof of Theorem 13.4. For z the centre of a lattice triangle in D and η a vector from z to one of the three neighbouring triangle centres, for $\alpha \in \{1, \tau, \tau^2\}$, the events $Q = Q_\alpha(z+\eta) \setminus Q_\alpha(z)$ and $\tilde{Q} = Q_{\tau\alpha}(z+\tau\eta) \setminus Q_{\tau\alpha}(z)$ have the same probability. To see this, label the vertices of the triangle at z by X, Y, Z, where X is opposite to η and we move anticlockwise around the triangle. Note that Q is the event that there exist disjoint black paths from Y to $a(\alpha\tau^2)a(\alpha)$ and from Z to $a(\alpha)a(\tau\alpha)$ and also a white path from X to $a(\alpha\tau)a(\alpha\tau^2)$. On the other hand, $\tilde{Q}$ is a similar event but where the path from Y must be white, and that from X must be black. To see that $\mathbb{P}(Q) = \mathbb{P}(\tilde{Q})$, explore the lattice from $a(\alpha)$ just as far as is needed to find suitable black paths (for Q) from Y and Z. Supposing this done, the conditional probability of the required white path from X is the same as if we required it to be black (and disjoint from the other paths). Hence Q and $\tilde{Q}$ both have the same probability as the event of three disjoint black paths to the required boundary segments.

Set $P_\alpha(z, \eta) = \mathbb{P}(Q)$. We have shown that

$$P_\alpha(z, \eta) = P_{\tau\alpha}(z, \tau\alpha). \tag{42}$$

This is a discrete version of the $2\pi/3$-Cauchy–Riemann equations for the triple $(H_1, H_\tau, H_{\tau^2})$. The rest of the proof is analytic. We accept here without proof the following results

Lemma 13.5 (Hölder estimate)**.** *There are constants $\varepsilon > 0$ and $C < \infty$, depending only on $(D, a(1), a(\tau), a(\tau^2))$, such that*

$$|H_\alpha(z) - H_\alpha(z')| \leqslant C(|z - z'| \wedge \delta)^\varepsilon.$$

Also, $H_\alpha(a(\alpha)) \to 1$ as $\delta \to 0$.

The proof uses a a classical method for regularity estimates in percolation due to Russo, Seymour and Welsh.

Lemma 13.6. *For any equilateral triangular contour Γ, of side length ℓ, interpolating neighbouring centres of lattice triangles, define the discrete contour integral*

$$\int_{\Gamma}^{\delta} H(z)dz = \delta \sum_{z\in A_1} H(z) + \delta\tau \sum_{z\in A_\tau} H(z) + \delta\tau^2 \sum_{z\in A_{\tau^2}} H(z),$$

where A_α is the set of centres along the side parallel to α. (Make some convention at the corners.) Then

$$\int_{\Gamma}^{\delta} H_\alpha(z)dz = \frac{1}{\tau}\int_{\Gamma}^{\delta} H_{\alpha\tau}(z)dz + O(\ell\delta^\varepsilon).$$

The proof is an elementary, if complicated, resummation argument, using the identity

$$H_\alpha(z+\eta) - H_\alpha(z) = P_\alpha(z,\eta) - P_\alpha(z+\eta,-\eta)$$

and, from the preceding lemma, the estimate $P_\alpha(z,\eta) \leqslant C\delta^\varepsilon$ for some stray terms.

The Hölder estimate implies that every sequence $\delta_n \downarrow 0$ contains a subsequence δ_{n_k} such that $H_\alpha^{\delta_{n_k}}$ converges uniformly on D for all α, and any such subsequential limits, h_α say, must have boundary values $h_\alpha(a(\alpha)) = 1$ and $h_\alpha(z) = 0$ on $a(\alpha\tau)a(\alpha\tau^2)$. Moreover, by Lemma 13.6, we must have

$$\int_\Gamma h_\alpha(z)dz = \frac{1}{\tau}\int_\Gamma h_{\alpha\tau}(z)dz.$$

Set $h = h_1 + (i/\sqrt{3})(h_\tau - h_{\tau^2})$, then

$$\int_\Gamma h(z)dz = 0$$

for all Γ, so h is holomorphic by Morera's theorem, and $h_\alpha = \mathrm{Re}(h/\alpha)$ is harmonic for all α. Hence we obtain

$$\nabla_\eta h_\alpha = \nabla_{\tau\alpha} h_{\tau\alpha}.$$

(This can be considered as the limiting form of the key observation on the discrete model (42), but the limit has not been justified directly.) This relation implies that the directional derivatives of h_1 on $a(\tau^2)a(1)$ and $a(\tau)a(1)$ at an angle τ to the tangent are zero. Thus we have a (conformally-invariant) Dirichlet-Neumann problem for h_1. In the case $D = \Delta$, the affine function f_1 is obviously a solution, and moreover it is the only solution. Hence the functions $H_1^\delta, H_\tau^\delta, H_{\tau^2}^\delta$ each have exactly one uniform limit point as $\delta \to 0$, given by h_1, h_τ, h_{τ^2} respectively, as required. $\square$

14 SLE(8/3) and restriction

14.1 Brownian excursion in the upper half-plane

Let $z = x + iy \in \bar{\mathbb{H}}$. Let $(X_t)_{t\geqslant 0}$ be a Brownian motion in $\mathbb{R}$, starting from x. Let $(W_t)_{t\geqslant 0}$ be a Brownian motion in $\mathbb{R}^3$, starting from $(y, 0, 0)$, and independent of $(X_t)_{t\geqslant 0}$. Set $R_t = |W_t|$. Then $(R_t)_{t\geqslant 0}$ is a Bessel process of dimension 3 starting from y. Set $E_t = X_t + iR_t$. The process $(E_t : t \geqslant 0)$ is called a *Brownian excursion in* $\mathbb{H}$ *starting from* z. Whilst this process is of interest in its own right, we introduce it here primarily as a means to study SLE(8/3), in particular using the following formula for the derivative of the mapping-out function. For a compact $\mathbb{H}$-hull A with $0 \notin \bar{A}$, we write ϕ_A for the shifted mapping-out function, given by

$$\phi_A(z) = g_A(z) - g_A(0).$$

Proposition 14.1. *Let A be a compact $\mathbb{H}$-hull with $0 \notin \bar{A}$. Let $(E_t)_{t\geqslant 0}$ be a Brownian excursion in $\mathbb{H}$ starting from 0. Then*

$$\mathbb{P}_0((E_t)_{t\geqslant 0} \textit{ does not hit } A) = \phi_A'(0).$$

Proof. Let $(Z_t)_{t\geqslant 0}$ be a complex Brownian motion starting from $z = x + iy \in \mathbb{H} \setminus A$. Let $(E_t)_{t\geqslant 0}$ be a Brownian excursion in $\mathbb{H}$, also starting from z. Write $Z_t = X_t + iY_t$. Define for $r \geqslant 0$

$$T_r = \inf\{t \geqslant 0 : Y_t = r\}, \quad T_A = \inf\{t \geqslant 0 : Z_t \in A\}, \quad S_r = \inf\{t \geqslant 0 : \operatorname{Im} E_t = r\}.$$

Fix $r > y$ and set $M_t = y^{-1}Y_{T_0\wedge T_r\wedge t}$. Then $(M_t)_{t\geqslant 0}$ is a bounded non-negative martingale with $M_0 = 1$ and with final value $Y_{T_0\wedge T_r} = (r/y)1_{\{T_0>T_r\}}$. Define a new probability measure $\tilde{\mathbb{P}}$ by

$$d\tilde{\mathbb{P}}/d\mathbb{P} = Y_{T_0\wedge T_r}.$$

Then $\tilde{\mathbb{P}}(T_0 > T_r) = 1$. Under $\tilde{\mathbb{P}}$, the processes $(X_t)_{t\geqslant 0}$ and $(Y_t)_{t\geqslant 0}$ remain independent and $(X_t)_{t\geqslant 0}$ is a Brownian motion. Consider the process $(B_t)_{t\geqslant 0}$ given by

$$dB_t = dY_t - M_t^{-1}dM_t dY_t = dY_t - 1_{\{t\leqslant T_r\}}Y_t^{-1}dt, \quad B_0 = 0.$$

Under $\tilde{\mathbb{P}}$, by Girsanov's theorem, $(B_t)_{t\geqslant 0}$ is a local martingale and hence, having the same quadratic variation as $(Y_t)_{t\geqslant 0}$, is a Brownian motion, by Lévy's characterization. The stochastic differential equation

$$d\tilde{Y}_t = dB_t + \tilde{Y}_t^{-1}dt, \quad \tilde{Y}_0 = y$$

has a unique strong solution $(\tilde{Y}_t)_{t\geqslant 0}$. Then $Y_t = \tilde{Y}_t$ for $t \leqslant T_r$. By the Yamada–Watanabe theorem $(\tilde{Y}_t)_{t\geqslant 0}$ under $\tilde{\mathbb{P}}$ has the same law as $(\operatorname{Im}(E_t))_{t\geqslant 0}$ under $\mathbb{P}$. So $(X_t + i\tilde{Y}_t)_{t\geqslant 0}$ under $\tilde{\mathbb{P}}$ has the same law as $(E_t)_{t\geqslant 0}$ under $\mathbb{P}$. Hence $(Z_t)_{t\leqslant T_r}$ under $\tilde{\mathbb{P}}$ has the same law as $(E_t)_{t\leqslant S_r}$ under $\mathbb{P}$. Set

$$p_r(z) = \mathbb{P}_z((E_t)_{t\leqslant S_r} \text{ does not hit } A).$$

Then

$$p_r(z) = \tilde{\mathbb{P}}_z((Z_t)_{t\leqslant T_r} \text{ does not hit } A) = \mathbb{E}_z(y^{-1}Y_{T_0\wedge T_r}1_{\{T_A>T_r\}}) = (r/y)\mathbb{P}_z(T_r < T_0 \wedge T_A).$$

Now $g_A(z) - z \to 0$ as $|z| \to \infty$ so, for r sufficiently large,

$$|\operatorname{Im} g_A(z) - r| \leqslant 1 \quad \text{whenever} \quad \operatorname{Im}(z) = r$$

and hence, by conformal invariance of Brownian motion,

$$\frac{\operatorname{Im} g_A(z)}{r+1} = \mathbb{P}_{g_A(z)}(T_{r+1} < T_0) \leqslant \mathbb{P}_z(T_r < T_0 \wedge T_A) \leqslant \mathbb{P}_{g_A(z)}(T_{r-1} < T_0) = \frac{\operatorname{Im} g_A(z)}{r-1}.$$

So

$$\mathbb{P}_z((E_t)_{t\geqslant 0} \text{ does not hit } A) = \lim_{r\to\infty} p_r(z) = \operatorname{Im} g_A(z)/y.$$

Note that $\operatorname{Im} g_A(z)/y \to g_A'(0) > 0$ as $z \to 0$ in $\mathbb{H}$. Take now $z = 0$, fix $\varepsilon > 0$ with $A \cap \varepsilon\mathbb{D} = \emptyset$, and set

$$S = \inf\{t \geqslant 0 : |E_t| = \varepsilon\},$$

then $|E_S| = \varepsilon$ and $\operatorname{Im} E_S > 0$ almost surely. Hence, by the strong Markov property of $(E_t)_{t\geqslant 0}$ and bounded convergence, as $\varepsilon \to 0$,

$$\mathbb{P}_0((E_t)_{t\geqslant 0} \text{ does not hit } A) = \mathbb{E}(\operatorname{Im} g_A(E_S)/\operatorname{Im}(E_S)) \to g_A'(0) = \phi_A'(0).$$

□

14.2 Restriction property of SLE(8/3)

We begin with a result for SLE(8/3) which is closely analogous to the result just proved for the Brownian excursion.

Proposition 14.2. *Let A be a compact $\mathbb{H}$-hull with $0 \notin \bar{A}$. Let $(\gamma_t)_{t\geqslant 0}$ be an SLE(8/3). Then*

$$\mathbb{P}((\gamma_t)_{t\geqslant 0} \textit{ does not hit } A) = \phi_A'(0)^{5/8}.$$

Proof. Set $K_t = \{\gamma_s : s \in (0,t]\}$ and $T = \inf\{t \geqslant 0 : \gamma_t \in A\}$. The Loewner transform $(\xi_t)_{t\geqslant 0}$ of $(K_t)_{t\geqslant 0}$ is a Brownian motion of diffusivity $\kappa = 8/3$. For $t < T$, set $\tilde{K}_t = \phi_A(K_t)$ and $\phi_t = g_{\tilde{K}_t} \circ \phi_A \circ (g_{K_t})^{-1}$. Then $\phi_t : \mathbb{H} \setminus g_{K_t}(A) \to \mathbb{H}$ is a conformal isomorphism and $\phi_t(z) - z + g_A(0) \to 0$ as $|z| \to \infty$, so ϕ_t is a shift of the mapping-out function for $g_{K_t}(A)$. Set $\Sigma_t = \phi_t'(\xi_t)$. The set $S_0 = \{(t,x) : t \in [0,T), x \in I_{K_t}\}$ is open in $[0,\infty)\times\mathbb{R}$ and $\xi_t \in I_{K_t}$ for all $t < T$. By Proposition 12.6, the adapted random map $(t,x) \mapsto \phi_t'(x) : S_0 \to \mathbb{R}$ is $C^{1,2}$ and

$$\dot{\phi}_t'(\xi_t) = \frac{1}{2}\frac{\phi_t''(\xi_t)^2}{\phi_t'(\xi_t)} - \frac{4}{3}\phi_t'''(\xi_t)$$

for all $t < T$. By the generalized Itô formula, we have

$$d\Sigma_t = \dot{\phi}'_t(\xi_t)dt + \phi''_t(\xi_t)d\xi_t + \frac{1}{2}\phi'''_t(\xi_t)d\xi_t d\xi_t.$$

Since $d\xi_t d\xi_t = \kappa dt$, this simplifies to give

$$d\Sigma_t = \phi''_t(\xi_t)d\xi_t + \frac{1}{2}\frac{\phi''_t(\xi_t)^2}{\phi'_t(\xi_t)}dt.$$

Fix $\alpha \in (0,1]$ and set $M_t = \Sigma_t^\alpha$. By Itô's formula,

$$dM_t = \alpha\Sigma_t^{\alpha-1}d\Sigma_t + \frac{1}{2}\alpha(\alpha-1)\Sigma_t^{\alpha-2}d\Sigma_t d\Sigma_t = \alpha M_t dY_t,$$

where

$$dY_t = \frac{d\Sigma_t}{\Sigma_t} + \frac{1}{2}(\alpha-1)\frac{\phi''_t(\xi_t)^2}{\Sigma_t^2}\kappa dt = \frac{\phi''_t(\xi_t)}{\Sigma_t}d\xi_t + \frac{1}{2}(1+(\alpha-1)\kappa)\frac{\phi''_t(\xi_t)^2}{\Sigma_t^2}dt.$$

We choose $\alpha = 5/8$ so the final term vanishes. Then $(Y_t)_{t<T}$ and hence also $(M_t)_{t<T}$ is a continuous local martingale.

By Proposition 14.1, conditional on γ, we have

$$\phi'_t(\xi_t) = \mathbb{P}_{\xi_t}((E_s)_{s\geqslant 0} \text{ does not hit } g_{K_t}(A)). \tag{43}$$

In particular $M_t \in [0,1]$ for all $t < T$, so M_t has an almost sure limit, M_T say, as $t \uparrow T$ and then by optional stopping

$$\mathbb{E}(M_T) = M_0 = \phi'_A(0)^{5/8}.$$

We shall show that $M_T = 1_{\{T=\infty\}}$ almost surely, so $\mathbb{P}(T=\infty) = \mathbb{E}(M_T) = \phi'_A(0)^{5/8}$, as required.

Consider first the case where $T = \infty$. We want to show that

$$\lim_{t\to\infty} \mathbb{P}_{\xi_t}((E_s)_{s\geqslant 0} \text{ hits } g_{K_t}(A)) = 0.$$

There exist connected compact $\mathbb{H}$-hulls A^- and A^+ such that $A \subseteq A^- \cup A^+$ and which $(\gamma_t)_{t\geqslant 0}$ does not hit. Hence we may reduce to the case where A is connected. By Propositions 6.2 and 6.1, we have

$$\operatorname{rad}(g_t(A)) \leqslant \operatorname{cap}(g_t(A)) \leqslant \operatorname{cap}(A) \leqslant 4\operatorname{rad}(A).$$

Fix $x \in \bar{A} \cap \mathbb{R}$. By Proposition 10.1, we have $|g_t(x) - \xi_t| \to \infty$ as $t \to \infty$. Hence, as $t \to \infty$,

$$\mathbb{P}_{\xi_t}((E_s)_{s\geqslant 0} \text{ hits } g_{K_t}(A)) \leqslant \mathbb{P}_0((E_s)_{s\geqslant 0} \text{ hits } g_{K_t}(x) - \xi_t + 8r\bar{\mathbb{D}}) \to 0.$$

Consider now the case where $T < \infty$. Write A_0 for the component of A containing γ_T and assume for now that the boundary of A_0 in $\mathbb{H}$ may be parametrized as a simple smooth curve $(\beta(u) : u \in \mathbb{R})$, with $\beta(0) = \gamma_T$. By symmetry it will suffice to consider the case where A_0 is based on $(0, \infty)$. Write A_0^o for the interior of A_0. Then

$$\lim_{t\uparrow T} \mathbb{P}_{\xi_t}((E_s)_{s\geqslant 0} \text{ hits } g_{K_t}(A_0)) \geqslant \mathbb{P}_{\xi_T}((E_s)_{s\geqslant 0} \text{ hits } g_{K_T}(A_0^o)).$$

By Proposition 16.4,

$$\liminf_{u\downarrow 0} \mathbb{P}_{\beta(u)}((B_s)_{s\geqslant 0} \text{ hits } \gamma(0,T] \text{ on the left side}) \geqslant 1/2.$$

Hence, by conformal invariance,

$$\liminf_{u\downarrow 0} \mathbb{P}_{g_T(\beta(u))}((B_s)_{s\geqslant 0} \text{ hits } \mathbb{R} \text{ to the left of } \xi_T) \geqslant 1/2.$$

and so, since $1/3 < 1/2$,

$$\liminf_{u\downarrow 0} \arg(g_T(\beta(u)) - \xi_T) \geqslant \pi/3.$$

Then

$$\mathbb{P}_{\xi_T}((E_s)_{s\geqslant 0} \text{ hits } g_{K_T}(A_0^o)) \geqslant \mathbb{P}_0(\Omega_0)$$

where

$$\Omega_0 = \cap_{n\in\mathbb{N}}\{\arg(E_s) \in (0, \pi/3) \text{ for some } s \in (0, 1/n)\}.$$

Recall the representation $E_s = X_s + i|W_s|$, where $(X_s)_{s\geqslant 0}$ and $(W_s)_{s\geqslant 0}$ are Brownian motions in $\mathbb{R}$ and $\mathbb{R}^3$ respectively. Then, by a scaling argument, $\mathbb{P}_0(\Omega_0) > 0$ and so $\mathbb{P}_0(\Omega_0) = 1$ by Blumenthal's zero-one law.

For general A, there is a sequence of compact $\mathbb{H}$-hulls $A_n \downarrow A$ such that the boundary in $\mathbb{H}$ of every component of every A_n is a simple smooth curve. Then, using Proposition 14.1,

$$\mathbb{P}((\gamma_t)_{t\geqslant 0} \text{ does not hit } A_n) = \mathbb{P}((E_t)_{t\geqslant 0} \text{ does not hit } A_n)^{5/8}$$

for all n. On letting $n \to \infty$ and using Proposition 14.1 again, we obtain the desired result for A. □

The proposition just proved leads by some general considerations to the following invariance property for SLE(8/3), called the *restriction property*. We defer the proof to the next section in order to put it in a more general context.

Theorem 14.3. *Let A be a compact $\mathbb{H}$-hull with $0 \notin \bar{A}$. Let $(\gamma_t)_{t\geqslant 0}$ be an SLE$(8/3)$. Then, conditional on the event $\{(\gamma_t)_{t\geqslant 0}$ does not hit $A\}$, the process $(\phi_A(\gamma_t))_{t\geqslant 0}$ in its canonical reparametrization is also an SLE$(8/3)$.*

Suppose α is a nonnegative integer and A is a compact $\mathbb{H}$-hull such that $0 \notin \bar{A}\cap\mathbb{R}$. Then $\Phi'_A(0)^\alpha$ is the probability that α independent Brownian excursions avoid A, by Proposition 14.1. Hence this is the probability that the hull generated by α independent Brownian excursions does not intersect A. Thus one way to informally interpret the result of Theorem 14.2 is to say that the SLE(8/3) chord can be thought of as 5/8 of a Brownian excursion. More precisely, we have the following result as an immediate corollary to Proposition 14.1 and Theorem 14.2:

Theorem 14.4. *The compact hull generated by 8 independent SLE(8/3) chords and the compact hull generated by 5 independent Brownian excursions have the same distribution.*

One of the particularly striking aspects of this result is that the curves themselves (SLE(8/3) and Brownian excursions) are very different from one another.

14.3 Restriction measures

We shift attention from compact $\mathbb{H}$-hulls to a different class of subsets in $\mathbb{H}$. A *filling* is any connected set K in $\mathbb{H}$ having 0 and ∞ as limit points in $\hat{\mathbb{H}}$ and such that $\mathbb{H} \setminus K$ is the union of two simply connected domains D^- and D^+ which are neighbourhoods of $(-\infty, 0)$ and $(0, \infty)$ in $\mathbb{H}$ respectively. Write S for the set of all such fillings. Write $\mathcal{N}$ for the set of simply connected domains which are neighbourhoods of both 0 and ∞ in $\mathbb{H}$. For $D \in \mathcal{N}$, define $S_D = \{K \in S : K \subseteq D\}$. Set $\mathcal{A} = \{S_D : D \in \mathcal{N}\}$ and write $\mathcal{S}$ for the σ-algebra on S generated by $\mathcal{A}$.

A random filling K (that is, an $(S, \mathcal{S})$-random variable) is said to have the *restriction property* if, for any $D \in \mathcal{N}$ and for $A = \mathbb{H} \setminus D$, the conditional distribution of the random filling $\phi_A(K)$ given $K \subseteq D$ is equal to the distribution of K. Since $\mathcal{A}$ is a π-system generating $\mathcal{S}$, it is equivalent that these distributions agree on $\mathcal{A}$, thus K has the restriction property if and only if, for all pairs $D, D' \in \mathcal{N}$ with $D' \subseteq D$, we have

$$\mathbb{P}(K \subseteq D') = \mathbb{P}(K \subseteq D)\mathbb{P}(K \subseteq \phi_A(D')).$$

The law of K on $(S, \mathcal{S})$ is then called a *restriction measure*.

Theorem 14.5. *Let $(\gamma_t)_{t \geqslant 0}$ be an SLE(8/3) and set $K = \{\gamma_t : t \in (0, \infty)\}$. Then K is a random filling and K has the restriction property.*

Proof. By Propositions 10.3, 11.3 and 11.5, K is a simple path in $\mathbb{H}$ with limit points 0 and ∞ in $\hat{\mathbb{H}}$. The sets $\{K \subseteq D\}$ for $D \in \mathcal{N}$ are all measurable. Hence K is a random filling.

For $D' \in \mathcal{N}$ with $D' \subseteq D$ and for $A = \mathbb{H} \setminus D$ and $A' = \mathbb{H} \setminus D'$ and $B = \mathbb{H} \setminus \phi_A(D')$, we have

$$\phi_{A'} = \phi_B \circ \phi_A.$$

Then, by Proposition 14.2, we have

$$\mathbb{P}(K \subseteq D') = \phi_{A'}'(0)^{5/8} = \phi_B'(0)^{5/8}\phi_A'(0)^{5/8} = \mathbb{P}(K \subseteq D)\mathbb{P}(K \subseteq \phi_A(D')).$$

Hence K has the restriction property. $\square$

Given a Brownian excursion $(E_t)_{t \geqslant 0}$, consider the set K which is the union of $K_0 = \{E_t : t \in (0, \infty)\}$ and all the bounded components of $\mathbb{H} \setminus K_0$. Then K is also a random filling and, by Proposition 14.1 and the same argument just used, K also has the restriction property. We have introduced restriction measures in order to consider SLE(8/3) in this context. Much more is known in general than we have discussed.

Proof of Theorem 14.3. Write S_0 for the set of fillings in $\mathbb{H}$ of the form $K = \{\gamma_t : t \in (0,\infty)\}$, where $(\gamma_t)_{t\geqslant 0}$ is a simple path in $\bar{\mathbb{H}}$ parametrized so that $\mathrm{hcap}(\gamma(0,t]) = 2t$ for all t. It is straightforward to see that S_0 is $\mathcal{S}$-measurable and we shall show also that the map $\theta : S_0 \to C([0,\infty),\bar{\mathbb{H}})$ given by $\theta(K) = \gamma$ is $\mathcal{S}$-measurable. For $(\gamma_t)_{t\geqslant 0}$ an SLE(8/3) and $K = \{\gamma_t : t \in (0,\infty)\}$, for A a compact $\mathbb{H}$-hull with $0 \notin \bar{A}$ and $D = \mathbb{H} \setminus A$, we have

$$\{(\gamma_t)_{t\geqslant 0} \text{ does not hit } A\} = \{K \subseteq D\}$$

and on this event the canonical reparametrization $(\tilde{\gamma}_t)_{t\geqslant 0}$ of $(\phi_A(\gamma_t))_{t\geqslant 0}$ is given by $\theta(\phi_A(K))$. Then, for B a measurable set in $C([0,\infty),\bar{\mathbb{H}})$, the set $\theta^{-1}(B)$ is $\mathcal{S}$-measurable and we obtain

$$\begin{aligned}&\mathbb{P}((\tilde{\gamma}_t)_{t\geqslant 0} \in B | \{(\gamma_t)_{t\geqslant 0} \text{ does not hit } A\}) \\ &\qquad = \mathbb{P}(\phi_A(K) \in \theta^{-1}(B) | K \subseteq D) = \mathbb{P}(K \in \theta^{-1}(B)) = \mathbb{P}((\gamma_t)_{t\geqslant 0} \in B).\end{aligned}$$

We now complete the proof of the theorem by showing the measurability of the map θ. For $n \geqslant 0$, write L_n for the dyadic lattice $2^{-n}\{j + ik : j \in \mathbb{Z},\, k \in \mathbb{Z}^+\}$. For each $p \in L_n$, consider the set $Q = Q(p) = p + \{x + iy : x, y \in [0, 2^{-n}]\}$ and write $\mathcal{N}_Q$ for the countable set of domains $D = \mathbb{H} \setminus A \in \mathcal{N}$ where $A = A^- \cup A^+$ and A^- and A^+ are disjoint simple paths which are unions of horizontal and vertical dyadic line segments, and which together with some boundary interval $I(D)$ of Q and some interval of $\mathbb{R}$ containing 0, form a simple closed curve in $\overline{\mathbb{H} \setminus Q}$. For $D \in \mathcal{N}_Q$, write $K(D)$ for the hull whose boundary in $\mathbb{H}$ consists of $A^- \cup A^+ \cup I(D)$. For $K \in S_0$, set $\gamma = \theta(K)$ and define

$$\tau_Q(K) = \inf\{t \geqslant 0 : \gamma_t \in Q\}, \quad h_Q(K) = \mathrm{hcap}(\gamma(0, \tau_Q(K)]), \quad e_Q(K) = \gamma_{\tau_Q(K)}.$$

By Proposition 6.5, given $t > 0$, we have $h_Q(K) < t$ if and only if $K \subseteq D$ for some $D \in \mathcal{N}_Q$ with $\mathrm{hcap}(K(D)) < t$. Also, given an open boundary interval I of Q, we have $e_Q(K) \in I$ if and only if $K \subseteq D$ for some $D \in \mathcal{N}_Q$ with $I(D) \subseteq I$. Hence $h_Q : S_0 \to [0,\infty]$ and $e_Q : S_0 \to \bar{\mathbb{H}}$ are both $\mathcal{S}$-measurable. For each $n \geqslant 0$, choose an enumeration $(p_m : m \geqslant 0)$ of L_n so that so that $h_m = h_{Q(p_m)}(K)$ is non-decreasing in m and set $e_m = e_{Q(p_m)}(K)$. Note that if $h_m = h_{m'}$ then $e_m = e_{m'}$. Set $h_m = 2t_m$. Define a path $(\theta_t^{(n)}(K))_{t\geqslant 0}$ by linear interpolation of $((t_m, e_m) : m \geqslant 0)$. Then $\theta^{(n)} : S_0 \to C([0,\infty),\bar{\mathbb{H}})$ is measurable for all n. Now $\theta_t(K) = \theta_t^{(n)}(K)$ for all $t \in T(n) = \{t_m : m \geqslant 0\}$ and, since $\theta(K)$ is simple, $\cup_n M(n)$ is dense in $[0,\infty)$. Hence, by uniform continuity, the paths $\theta^{(n)}(K)$ converge to $\theta(K)$ uniformly on compacts, so θ is also measurable, as required. $\square$

15 SLE(4) and the Gaussian free field

We define the planar Gaussian free field and review some of its properties. Then we prove a relation with SLE(4) due to Schramm and Sheffield which suggests that SLE(4) can be interpreted as a fracture line of this Gaussian process. Since the free field is distribution-valued, we begin with a quick review of some classical material on function spaces and distributions.

15.1 Conformal invariance of function spaces

Let D be a domain. A *test-function* on D is an infinitely differentiable function on D which is supported on some compact subset of D. The set of all such test-functions is denoted $\mathcal{D}(D)$. The set $\mathcal{D}(D)$ is made into a locally convex topological vector space[12] in which convergence is characterized as follows. A sequence $f_n \to 0$ in $\mathcal{D}(D)$ if and only if there is a compact set $K \subseteq D$ such that $\operatorname{supp} f_n \subseteq K$ for all n and f_n and all its derivatives converge to 0 uniformly on D. A continuous linear map $u : \mathcal{D}(D) \to \mathbb{R}$ is called a *distribution*[13] on D. Thus, the set of distributions on D is the dual space of $\mathcal{D}(D)$. It is denoted by $\mathcal{D}'(D)$ and is given the weak-$*$ topology. Thus $u_n \to u$ in $\mathcal{D}'(D)$ if and only if $u_n(\rho) \to u(\rho)$ for all $\rho \in \mathcal{D}(D)$. In this context, we think of each $\rho \in \mathcal{D}(D)$ as specifying a suitably regular signed measure on D, given by $\rho(x)dx$, and of each $u \in \mathcal{D}'(D)$ as a generalized function on D, which can be viewed through its 'averages' $u(\rho)$ with respect to test-functions. We will freely identify ρ with the signed measure $\rho(x)dx$. Note that, since $\mathcal{D}(D)$ is separable, the Borel σ-algebra on $\mathcal{D}'(D)$ is generated by the coordinate functions $u \mapsto u(\rho)$. We specialize from this point on to the planar case, where we note the following result. The proof, which is left as an exercise, rests on the fact that ϕ and all its derivatives are bounded on compact subsets of D_0, and ϕ^{-1} and all its derivatives are bounded on compact subsets of D.

Proposition 15.1. *Let $\phi : D_0 \to D$ be a conformal isomorphism of planar domains. The map $f \mapsto f \circ \phi^{-1}$ is a linear homeomorphism $\mathcal{D}(D_0) \to \mathcal{D}(D)$.*

For $\rho \in \mathcal{D}(D)$, the image measure of $\rho(x)dx$ by ϕ^{-1} is given by $\rho_0(x)dx$, where $\rho_0 = (\rho \circ \phi)|\phi'|^2$. The map $\rho \mapsto \rho_0$ is a linear homeomorphism $\mathcal{D}(D) \to \mathcal{D}(D_0)$.

For $u_0 \in \mathcal{D}'(D_0)$, consider the distribution u on D given by $u(\rho) = u_0(\rho_0)$. We write formally $u = u_0 \circ \phi^{-1}$. The map $u_0 \mapsto u$ is a linear homeomorphism $\mathcal{D}'(D_0) \to \mathcal{D}'(D)$.

Let D be a proper simply connected domain. The Green function $(G_D(x, y) : x, y \in D)$ was introduced in Section 3.3. Write $\mathcal{M}_D$ for the set of Borel measures μ on D having finite energy

$$\mathcal{E}_D(\mu) = \int_{D^2} G_D(x, y)\mu(dx)\mu(dy).$$

The energy has a conformal invariance property which it inherits from the Green function. The proof is left as an exercise.

[12] See [6] for more details.

[13] This conflicts with the usage of distribution to mean the law of a random variable but is standard and should not cause confusion.

Proposition 15.2. *Let $\phi : D_0 \to D$ be a conformal isomorphism of proper simply connected domains and let μ_0 be a Borel measure on D_0. Set $\mu = \mu_0 \circ \phi^{-1}$. Then*

$$\mathcal{E}_D(\mu) = \mathcal{E}_{D_0}(\mu_0).$$

Consider now a bounded domain D and define for $f, g \in \mathcal{D}(D)$

$$\langle f, g\rangle_{H^1(D)} = \frac{1}{2}\int_D \langle \nabla f, \nabla g\rangle dx.$$

Then $\langle ., .\rangle_{H^1(D)}$ is an inner product on $\mathcal{D}(D)$. Write $\|.\|_{H^1(D)}$ for the associated norm. Choose $R \in (0, \infty)$ so that $|x| \leqslant R$ for all $x \in D$. For $f \in \mathcal{D}(D)$ and for $r \in [0, R]$ and $|z| = 1$, we have

$$|f(rz)|^2 = \left|\int_r^R \langle z, \nabla f(tz)\rangle dt\right|^2 \leqslant \int_r^R |\nabla f(tz)|^2 dt.$$

So, by Fubini's theorem, we obtain Poincaré's inequality

$$\|f\|^2_{L^2(D)} = \int_D |f|^2 dx = \int_0^{2\pi}\int_0^R |f(re^{i\theta})|^2 r dr d\theta \leqslant \frac{R}{2}\int_D |\nabla f|^2 dx = R\|f\|^2_{H^1(D)}.$$

Write $\mathcal{L}^2(D)$ for the Hilbert space of square integrable functions on D, modulo almost everywhere equality. By Poincaré's inequality, for D bounded, we can complete $\mathcal{D}(D)$ within $\mathcal{L}^2(D)$ using $\|.\|_{H^1(D)}$ to obtain a Hilbert space, which is denoted $H^1_0(D)$. The set of distributions $\gamma \in \mathcal{D}'(D)$ such that $|\gamma(\rho)| \leqslant C\|\rho\|_{H^1}$ for all $\rho \in \mathcal{D}(D)$ is denoted $H^{-1}(D)$.

Proposition 15.3. *Let $\phi : D_0 \to D$ be a conformal isomorphism of bounded planar domains. Then the map $f \mapsto f \circ \phi^{-1}$ is a Hilbert space isomorphism $H^1_0(D_0) \to H^1_0(D)$.*

Proof. Set $\psi = \phi^{-1}$. For $f \in \mathcal{D}(D_0)$, by the Jacobian formula, we have

$$\|f \circ \psi\|^2_{H^1(D)} = \int_D (|\nabla f|^2 \circ \psi)|\psi'|^2 dx = \int_{D_0} |\nabla f|^2 dx = \|f\|^2_{H^1(D_0)}.$$

Since $\mathcal{D}(D_0)$ is dense in $H^1_0(D_0)$, this isometry property extends to the whole space. □

We will use the following basic fact from partial differential equations. There is a complete orthonormal system $(f_n : n \in \mathbb{N})$ in $H^1_0(D)$ and a non-decreasing sequence $(\lambda_n : n \in \mathbb{N})$ in $(0, \infty)$ such that $f_n \in C^\infty(D)$ and $-\frac{1}{2}\Delta f_n = \lambda_n f_n$ for all n. See for example [3]. Set $e_n = \lambda_n^{1/2} f_n$. Then $(e_n : n \in \mathbb{N})$ is a complete orthonormal system in $L^2(D)$. To see this we use the fact that $\mathcal{D}(D)$ is dense in $L^2(D)$. We defined the heat kernel p_D and the Green function G_D probabilistically in Section 3.3. So we have to work a little to make the following connection with the spectrum of the Laplacian.

Proposition 15.4. *Let D be a bounded planar domain. For all $n \in \mathbb{N}$, $x \in D$ and $t \geqslant 0$, we have*

$$f_n(x) = e^{\lambda_n t} \int_D p_D(t, x, y) f_n(y) dy = \lambda_n \int_D G_D(x, y) f_n(y) dy.$$

Proof. The second equality follows from the first by integration. By scaling, it will suffice to consider the case where $D \subseteq (0,1)^2$. Then there exists a $\mathbb{Z}^2$-periodic function f which agrees with f_n on D and vanishes on $(0,1]^2 \setminus D$, with $f \in L^2(\mathbb{T}) \cap H^1(\mathbb{T})$. Set

$$F(x) = \mathbb{E}_x \left(\sup_{s \leqslant t} |f(B_s)|^2 \right), \quad p(x) = \mathbb{P}_x(f(B_t) \to 0 \text{ as } t \to T(D)).$$

Then, by Lemma 16.3,

$$\int_{\mathbb{T}} F(x) dx < \infty, \quad \int_{\mathbb{T}} p(x) dx = 1.$$

Since p is harmonic in D, it is continuous, so $p(x) = 1$ for all $x \in D$. We will show that also $F(x) < \infty$ for all $x \in D$. Set $D_0 = \{y : |x - y| < \delta\}$ and choose $\delta > 0$ so that $D_0 \subseteq D$. Now f is bounded on D_0 so it will suffice to show that $\mathbb{E}_x(F(B_{T(D_0)}) < \infty$. For some $y \in D_0$, we have $F(y) < \infty$ so $\mathbb{E}_y(F(B_{T(D_0)}) < \infty$. But the hitting density from x on ∂D_0 is constant and that from y is bounded below. Hence $F(x) < \infty$.

Define

$$M_t = e^{\lambda_n t} f_n(B_t) 1_{\{t < T(D)\}}, \quad t \geqslant 0.$$

By Itô's formula, $(M_t : t < T(D))$ is a continuous local martingale. We have shown further that $M_t \to 0$ as $t \to T(D)$ and $\mathbb{E}_x(\sup_{s \leqslant t} |M_s|^2) < \infty$ for all $t \geqslant 0$. Hence $(M_t)_{t \geqslant 0}$ is a continuous martingale. The desired equality then follows by optional stopping:

$$f_n(x) = \mathbb{E}_x(M_0) = \mathbb{E}_x(M_{t \wedge T(D)}) = e^{\lambda_n t} \mathbb{E}_x \left(f_n(B_t) 1_{\{t < T(D)\}} \right) = e^{\lambda_n t} \int_D p_D(t, x, y) f_n(y) dy.$$

□

For all $t > 0$ and $x, y \in D$, by the Markov property, we have

$$p_D(t, x, y) = \int_D p_D(t/2, x, z) p(t/2, z, y) dz.$$

Since p_D is symmetric and $p_D(t, x, x) < \infty$, we see that $p_D(t/2, x, .) \in L^2(D)$. By Proposition 15.4,

$$\int_D p(t, x, y) e_n(y) dy = e^{-\lambda_n t} e_n(x)$$

so, by Parseval's identity,

$$p_D(t, x, y) = \sum_n e^{-\lambda_n t} e_n(x) e_n(y).$$

In particular, we deduce that

$$\int_D p_D(t,x,x)dx = \sum_n e^{-\lambda_n t}$$

and so, by Fubini, for $\alpha > 0$,

$$\int_0^\infty \int_D t^{\alpha-1} p_D(t,x,x)dxdt = \sum_n \int_0^\infty t^{\alpha-1} e^{\lambda_n t} dt = \Gamma(\alpha) \sum_n \lambda_n^{-\alpha}.$$

We note that the left side of this identity is increasing in the domain D. For $D = (0,\pi)^2$ the eigenfunctions $(f_n : n \in \mathbb{N})$ are given explicitly, after normalization and reordering, by $\sin mx \sin ny$ for $x, y \in (0,\pi)$ and $n, m \in \mathbb{N}$. Hence, for any domain $D \subseteq (0,\pi)^2$ and all $\alpha > 1$, we have

$$\sum_n \lambda_n^{-\alpha} \leqslant \sum_{m,n\in\mathbb{N}} (m^2+n^2)^{-\alpha} < \infty.$$

Proposition 15.5. *Let D be a bounded planar domain. Let $\mu \in \mathcal{M}_D$ and let $f \in H_0^1(D)$. Then f is integrable with respect to μ and the map $f \mapsto \mu(f)$ is a continuous linear functional on $H_0^1(D)$ of norm $\mathcal{E}_D(\mu)^{1/2}$.*

Proof. The iden We have

$$\mathcal{E}_D(\mu) = \int_{D\times D} \sum_n f_n(x) f_n(y) \mu(dx)\mu(dy) = \sum_n \mu(f_n)^2.$$

Then, for $f = \sum_n a_n f_n \in H_0^1(D)$, by Cauchy–Schwarz,

$$\mu(f) = \sum_n a_n \mu(f_n) \leqslant \left(\sum_n a_n^2\right)^{1/2} \left(\sum_n \mu(f_n)^2\right)^{1/2} = \|f\|_{H^1(D)} \mathcal{E}_D(\mu)^{1/2}.$$

□

15.2 Gaussian free field

Let D be a Greenian planar domain. A random variable Γ in $\mathcal{D}'(D)$ is said to be a *Gaussian free field in D with zero boundary values* if $\Gamma(\rho)$ is a Gaussian random variable with mean zero and variance $\mathcal{E}_D(\rho)$ for all $\rho \in \mathcal{D}(D)$. In the case where D is simply connected, recall from Section 1.3 the notion of the Martin boundary δD. Then, given a bounded measurable function f on δD, we say that a random variable Γ in $\mathcal{D}'(D)$ is a *Gaussian free field on D with boundary value f* if $\Gamma = \Gamma_0 + u$ for some Gaussian free field Γ_0 on D with zero boundary values, where u is the harmonic extension of f in D. Thus $\Gamma(\rho)$ is a Gaussian random variable of mean

$$\mathcal{H}_D(f,\rho) = \int_{\delta D\times D} f(y) h_D(x,dy)\rho(x)dx$$

and variance $\mathcal{E}_D(\rho)$ for all $\rho \in \mathcal{D}(D)$. Note that Γ cannot simply be evaluated at a given point in the boundary, nor anywhere where else for that matter.

Theorem 15.6. *Let D be a bounded planar domain. There exists a unique Borel probability measure on $\mathcal{D}'(D)$ which is the law of a Gaussian free field on D with zero boundary values.*

Proof. Let $(X_n : n \in \mathbb{N})$ be a sequence of independent standard Gaussian random variables. Set $S = \sum_n \lambda_n^{-2} X_n^2$ and define $\Omega_0 = \{S < \infty\}$. Then $\mathbb{E}(S) = \sum_n \lambda_n^{-2} < \infty$, so $\mathbb{P}(\Omega_0) = 1$. Fix $\rho \in \mathcal{D}(D)$ and set $a_n = \int_D f_n \rho dx$. Then

$$\sum_n \lambda_n^2 a_n^2 = \sum_n \langle \rho, f_n \rangle_{H^1(D)}^2 = \|\rho\|_{H^1(D)}^2 < \infty, \quad \sum_n a_n^2 = \mathcal{E}_D(\rho)$$

and $\|\rho\|_{H^1(D)} \to 0$ as $\rho \to 0$ in $\mathcal{D}(D)$. Define

$$\Gamma(\rho) = 1_{\Omega_0} \sum_n a_n X_n.$$

By Cauchy–Schwarz, the series converges absolutely on Ω_0 with

$$|\Gamma(\rho)| \leqslant \sqrt{S} \|\rho\|_{H^1(D)},$$

so $\Gamma = (\Gamma(\rho) : \rho \in \mathcal{D}(D))$ is a random variable in $\mathcal{D}'(D)$. The series converges also in L^2, so $\Gamma(\rho)$ is Gaussian, of mean zero and variance $\mathcal{E}_D(\rho)$. Hence Γ is a Gaussian free field on D with zero boundary values, as required.

On the other hand, for any Gaussian free field Γ on D with zero boundary values and for any $\rho = (\rho_1, \dots, \rho_k) \in \mathcal{D}(D)^k$, the characteristic function ϕ of the the random variable $\Gamma(\rho) = (\Gamma(\rho_1), \dots, \Gamma(\rho_k))$ in $\mathbb{R}^k$ is given by $\phi(\alpha) = \exp\{-\mathcal{E}_D(\bar{\rho})/2\}$, where $\bar{\rho} = \alpha_1 \rho_1 + \cdots + \alpha_k \rho_k \in \mathcal{D}(D)$ and $\alpha = (\alpha_1, \dots, \alpha_k)$. This determines uniquely the law of $\Gamma(\rho)$ on $\mathbb{R}^k$. Since the Borel σ-algebra on $\mathcal{D}'(D)$ is generated by the coordinate functions $u \mapsto u(\rho)$, this further determines uniquely the law of Γ on $\mathcal{D}'(D)$. □

The Gaussian free field inherits from the energy and harmonic measure a property of conformal invariance. The proof is left as an exercise.

Proposition 15.7. *Let $\phi : D \to D'$ be a conformal isomorphism of Greenian domains. Suppose that Γ is a Gaussian free field on D with zero boundary values. Then $\Gamma \circ \phi^{-1}$ is a Gaussian free field on D' with zero boundary values.*

Suppose now that D is simply connected, and that Γ is a Gaussian free field on D with boundary value f. Then $\Gamma \circ \phi^{-1}$ is a Gaussian free field on D' with boundary value $f \circ \phi^{-1}$.

As a corollary of this result and the Riemann mapping theorem, we see that the conclusion of Theorem 15.6 remains valid under the hypothesis that D is the image of a bounded domain under a conformal isomorphism, in particular whenever D is contained in a proper simply connected domain.

We will prove two useful extension properties of Gaussian free fields. First, they can be evaluated through less regular averages, in particular with respect to any measure of finite energy. Write $\mathcal{L}^2$ for the Hilbert space of all square-integrable random variables modulo almost sure equality. Write $\mathcal{G}_0$ for the closed subspace in $\mathcal{L}^2$ of zero-mean Gaussian random variables.

Proposition 15.8. *Let D be a bounded planar domain and let Γ be a Gaussian free field on D with zero boundary values. There is a unique Hilbert space isometry $\tilde{\Gamma} : H^{-1}(D) \to \mathcal{G}_0$ such that $\tilde{\Gamma}(\rho) = [\Gamma(\rho)]$ for all $\rho \in \mathcal{D}(D)$.*

Proof. Note that $[\Gamma(\rho)] \in \mathcal{G}_0$ for all $\rho \in \mathcal{D}(D)$. Consider the space of test-functions $\mathcal{D}(D)$ as a subspace of $H^{-1}(D)$ and note that $\mathcal{E}_D(\rho) = \|\rho\|^2_{H^{-1}(D)}$. Then $\mathcal{D}(D)$ is dense in $H^{-1}(D)$ and the map $\rho \mapsto [\Gamma(\rho)] : \mathcal{D}(D) \to \mathcal{G}_0$ is an isometry, which therefore extends uniquely to an isometry $\tilde{\Gamma} : H^{-1}(D) \to \mathcal{G}_0$. □

Note that, while we can regard $\tilde{\Gamma}(u)$ as a random variable for any $u \in H^{-1}(D)$, by a choice of representative, there is no guarantee of a regular version of these random variables as u varies, in contrast to the family $(\Gamma(\rho) : \rho \in \mathcal{D}(D))$ which we know belongs to $\mathcal{D}'(D)$ almost surely. Note also that, for D bounded and for $u \in H^{-1}(D)$, we have $u = \sum_n \lambda_n u(f_n) f_n$ in $H^{-1}(D)$, where $(f_n : n \in \mathbb{N})$ and $(\lambda_n : n \in \mathbb{N})$ are as in Section 15.1. Hence, by Doob's L^2 martingale convergence theorem,

$$\tilde{\Gamma}(u) = \sum_n u(f_n) Y_n \quad \text{a.s.}$$

where $(Y_n : n \in \mathbb{N})$ is the sequence of independent standard Gaussian random variables given by $Y_n = \lambda_n \tilde{\Gamma}(f_n)$. In particular, if Γ is constructed as in the proof of Theorem 15.6, then $Y_n = X_n$ for all n almost surely.

Second, the free field may be regarded as a distribution on a larger domain.

Proposition 15.9. *Let D be a subdomain of a proper simply connected domain D^*. Let Γ be a Gaussian free field on D with zero boundary values. Then Γ extends uniquely almost surely to a random variable $\bar{\Gamma}$ in $\mathcal{D}'(D^*)$ such that $\bar{\Gamma}(\rho)$ is a Gaussian random variable of mean zero and variance $\mathcal{E}_D(\rho)$ for all $\rho \in \mathcal{D}(D^*)$.*

Proof. By the Riemann mapping theorem and conformal invariance, we reduce to the case where $D^* = \mathbb{D}$. We make a variation of the proof of Theorem 15.6. Write $\bar{f}_n$ for the extension of f_n to $\mathbb{D}$ which vanishes identically on $\mathbb{D} \setminus D$. Then $(\bar{f}_n : n \in \mathbb{N})$ is an orthonormal system in $H^1_0(\mathbb{D})$. For $\rho \in H^1_0(\mathbb{D})$, set $\rho_n = \int_D f_n \rho dx = \int_{\mathbb{D}} \bar{f}_n \rho dx$. Then

$$\lambda_n \rho_n = \langle \rho, \bar{f}_n \rangle_{H^1(\mathbb{D})}, \quad \sum_n \lambda_n^2 \rho_n^2 \leqslant \|\rho\|^2_{H^1(\mathbb{D})}, \quad \sum_n \rho_n^2 = \mathcal{E}_D(\rho).$$

Set $\Omega_0 = \{\sum_n \lambda_n^{-2} Y_n^2 < \infty\}$ and define

$$\bar{\Gamma}(\rho) = 1_{\Omega_0} \sum_n \rho_n Y_n.$$

The same argument used in the proof of Theorem 15.6 now shows that $\bar{\Gamma} = (\bar{\Gamma}(\rho) : \rho \in \mathcal{D}'(\mathbb{D}))$ is a random variable in $\mathcal{D}'(\mathbb{D})$ with the required properties. For any other such random variable, Γ^* say, we have $\Gamma^*(\rho) = \tilde{\Gamma}(\rho 1_D) = \bar{\Gamma}(\rho)$ almost surely for all $\rho \in \mathcal{D}'(\mathbb{D})$, so $\Gamma^* = \bar{\Gamma}$ almost surely on $\mathcal{D}'(\mathbb{D})$ by continuity. □

The Gaussian free field has the following Markov property.

Proposition 15.10. *Let D be a subdomain of a proper simply connected domain D^*. Let Γ^* be a Gaussian free field on D^*. Then Γ^* has an almost surely unique decomposition $\Gamma^* = \bar{\Gamma} + \Phi$ as a sum of independent random variables in $\mathcal{D}'(D^*)$ such that Φ is harmonic on D and $\bar{\Gamma}$ restricts to a Gaussian free field on D with zero boundary values.*

Proof. We reduce to the case where Γ^* has zero boundary values by subtracting if necessary the harmonic extension of its boundary values in D^*. We proceed as in the proof of Proposition 15.9, with $D^* = \mathbb{D}$, but with Y_n replaced by $Y_n^* = \tilde{\Gamma}^*(-\frac{1}{2}\Delta \bar{f}_n)$. Thus we define a random variable $\bar{\Gamma}$ in $\mathcal{D}'(\mathbb{D})$ by

$$\bar{\Gamma}(\rho) = \sum_n \rho_n Y_n^*$$

and $\bar{\Gamma}$ restricts to a Gaussian free field on D with zero boundary values. Set $\Phi = \Gamma^* - \bar{\Gamma}$. For $\rho \in \mathcal{D}(D)$, we have $\rho = \sum_n \lambda_n \rho_n \bar{f}_n$ in $H_0^1(\mathbb{D})$, so $\Delta\rho = \sum_n \lambda_n \rho_n \Delta \bar{f}_n$ in $H^{-1}(\mathbb{D})$. Also $\int_{\mathbb{D}}(-\frac{1}{2}\Delta\rho)\bar{f}_n dx = \lambda_n \rho_n$ by integration by parts. Hence, almost surely,

$$\bar{\Gamma}(-\tfrac{1}{2}\Delta\rho) = \sum_n \lambda_n \rho_n Y_n^* = \Gamma^*(-\tfrac{1}{2}\Delta\rho).$$

Then, by continuity, $\Phi(\Delta\rho) = 0$ for all $\rho \in \mathcal{D}(D)$ almost surely, so Φ is harmonic in D. Finally, for $\rho^* \in \mathcal{D}(\mathbb{D})$, we have $\rho^* = \Delta f$ for some $f \in H_0^1(\mathbb{D})$ and then $f = \rho_0 + h$ with $f \in H_0^1(D)$ and $h \in H_0^1(\mathbb{D})$ with $\Delta h = 0$ on D. Then $\Phi(\rho^*) = \tilde{\Gamma}^*(\Delta h)$ and $\mathbb{E}(\tilde{\Gamma}^*(\Delta h) Y_n^*) = 4\langle h, \bar{f}_n\rangle_{H^1(\mathbb{D})} = 0$ for all n. The Gaussian random variables Φ and $\bar{\Gamma}$ are then orthogonal and thus independent. □

15.3 Angle martingales for SLE(4)

We study a family of martingales for SLE(4) and their relation to the Green function. Then by integrating with respect to a test-function we obtain a splitting identity for the characteristic function of a certain Gaussian free field in $\mathbb{H}$.

Define s_0 on $\delta\mathbb{H}$ by $s_0(\pm x) = \pm 1$ for $x \in (0, \infty)$ and $s_0(0) = s_0(\infty) = 0$. Write σ_0 for the harmonic extension of s_0 in $\mathbb{H}$. Then

$$\sigma_0(z) = \int_{\delta\mathbb{H}} s_0(x) h_{\mathbb{H}}(z, dx) = 1 - (2/\pi)\arg(z), \quad z \in \mathbb{H}.$$

Let γ be an SLE(4). Write $(g_t(z) : z \in \mathbb{H}, t < \zeta(z))$ and $(\xi_t)_{t\geqslant 0}$ for the associated Loewner flow and Loewner tranform and set $H_t = \{z \in \mathbb{H} : t < \zeta(z)\}$. Define $s_t(x) = s_0(g_t(x) - \xi_t)$ for $x \in \delta H_t$. The harmonic extension σ_t of s_t in H_t is then given by $\sigma_t(z) = \sigma_0(g_t(z) - \xi_t)$.

Proposition 15.11. *For all $z \in \mathbb{H}$, the process $(\sigma_t(z) : t < \zeta(z))$ is a continuous local martingale and $\zeta(z) = \infty$ almost surely. Moreover, for all $w \in \mathbb{H} \setminus \{z\}$, the process $(\sigma_t(z)\sigma_t(w) + (4/\pi)G_{H_t}(z, w) : t < \zeta(z) \wedge \zeta(w))$ is also a continuous local martingale.*

Proof. Write $Z_t = g_t(z) - \xi_t$ and $W_t = g_t(w) - \xi_t$. From Loewner's equation, we have $dZ_t = (2/Z_t)dt - d\xi_t$ for $t < \zeta(z)$. Then, by Itô's formula,

$$d \log Z_t = \frac{dZ_t}{Z_t} - \frac{d[Z]_t}{2Z_t^2} = -\frac{d\xi_t}{Z_t} + \left(2 - \frac{\kappa}{2}\right)\frac{dt}{Z_t^2}.$$

Since $\kappa = 4$, this shows that the real and imaginary parts of $(\log Z_t : t < \zeta(z))$ are continuous local martingales. Now $Z_t \to 0$ as $t \to \zeta(z)$ when $\zeta(z) < \infty$, so $\log |Z_t| = \operatorname{Re} \log Z_t \to -\infty$. This is impossible for a continuous local martingale, so $\zeta(z) = \infty$ almost surely. Also $\sigma_t(z) = 1 - (2/\pi) \operatorname{Im} \log Z_t$, so $(\sigma_t(z) : t < \zeta(z))$ is a continuous local martingale, with

$$d\sigma_t(z) = \frac{2}{\pi} \operatorname{Im}\left(\frac{1}{Z_t}\right) d\xi_t.$$

Then, for $t < \zeta(z) \wedge \zeta(w)$,

$$d(\sigma_t(z)\sigma_t(w)) = dN_t + \frac{16}{\pi^2} \operatorname{Im}\left(\frac{1}{Z_t}\right) \operatorname{Im}\left(\frac{1}{W_t}\right) dt$$

for a continuous local martingale $(N_t : t < \zeta(z) \wedge \zeta(w))$. On the other hand, by conformal invariance of the Green function, for $t < \zeta(z) \wedge \zeta(w)$,

$$G_{H_t}(z, w) = G_{\mathbb{H}}(g_t(z), g_t(w)) = \frac{1}{\pi} \log \left| \frac{Z_t - \bar{W}_t}{Z_t - W_t} \right|.$$

Now $d(Z_t - W_t) = 2(Z_t - W_t)dt/(Z_t W_t)$, so

$$d \log(Z_t - W_t) = \frac{2dt}{Z_t W_t}, \quad d \log(Z_t - \bar{W}_t) = \frac{2dt}{Z_t \bar{W}_t}$$

so

$$\begin{aligned} dG_{H_t}(z, w) &= d \operatorname{Re}\left(\frac{1}{\pi} \log\left(\frac{Z_t - \bar{W}_t}{Z_t - W_t}\right)\right) \\ &= \operatorname{Re}\left(\frac{2}{\pi Z_t}\left(\frac{1}{\bar{W}_t} - \frac{1}{W_t}\right)\right) dt = -\frac{4}{\pi} \operatorname{Im}\left(\frac{1}{Z_t}\right) \operatorname{Im}\left(\frac{1}{W_t}\right) dt. \end{aligned}$$

Hence $d(\sigma_t(z)\sigma_t(w)) + (4/\pi)dG_{H_t}(z, w) = dN_t$ and the result follows. □

Proposition 15.12. *Set $\lambda = \sqrt{\pi/4}$. Write D^- and D^+ for the left and right components of $\mathbb{H} \setminus \gamma^*$. Then, for all $\rho \in \mathcal{D}(\mathbb{H})$, we have*

$$\exp\left\{i\lambda \mathcal{H}_{\mathbb{H}}(s_0, \rho) - \frac{\mathcal{E}_{\mathbb{H}}(\rho)}{2}\right\} = \mathbb{E}\left(\exp\left\{i\lambda\rho(D^+) - \frac{\mathcal{E}_{D^+}(\rho)}{2}\right\} \exp\left\{-i\lambda\rho(D^-) - \frac{\mathcal{E}_{D^-}(\rho)}{2}\right\}\right).$$

Proof. Fix $\rho \in \mathcal{D}(\mathbb{H})$ and set

$$M_t = \int_{\mathbb{H}\setminus\gamma^*} \lambda\sigma_t(z)\rho(z)dz.$$

For $z \in \mathbb{H}\setminus\gamma^*$, the map $t \mapsto \sigma_t(z)$ is continuous on $[0,\infty)$ and $|\sigma_t(z)| \leqslant 1$ so, by dominated convergence, $t \mapsto M_t$ is continuous on $[0,\infty)$, almost surely. We know that $\gamma_t \to \infty$ as $t \to \infty$ almost surely, so $\sigma_t \to \pm 1$ on $D^\pm$, and so

$$M_t \to \lambda\rho(D^+) - \lambda\rho(D^-).$$

By Fubini's theorem, the trace $\gamma^* = \{z \in \mathbb{H} : \zeta(z) < \infty\}$ has zero planar Lebesgue measure almost surely. Also, $G_{H_t} \to G_{D^\pm}$ on $D^\pm \times D^\pm$ and $G_{H_t} \to 0$ on $D^\pm \times D^\mp$ almost surely, so

$$\mathcal{E}_{H_t}(\rho) \to \mathcal{E}_{D^-}(\rho) + \mathcal{E}_{D^+}(\rho).$$

For all $z \in \mathbb{H}$, we have $\zeta(z) = \infty$ almost surely. The local martingales identified in Proposition 15.11 are uniformly bounded and so are true martingales. So, by Fubini's theorem, for $s \leqslant t$ and $A \in \mathcal{F}_s$,

$$\mathbb{E}(M_t 1_A) = \int_{\mathbb{H}} \mathbb{E}(\lambda\sigma_t(z)1_A)\rho(z)dz = \int_{\mathbb{H}} \mathbb{E}(\lambda\sigma_s(z)1_A)\rho(z)dz = \mathbb{E}(M_s 1_A).$$

and

$$\begin{aligned}\mathbb{E}((M_t^2 + \mathcal{E}_{H_t}(\rho))1_A) &= \int_{\mathbb{H}^2} \mathbb{E}((\lambda^2\sigma_t(z)\sigma_t(w) + G_{H_t}(z,w))1_A)\rho(z)\rho(w)dzdw \\ &= \int_{\mathbb{H}^2} \mathbb{E}((\lambda^2\sigma_s(z)\sigma_s(w) + G_{H_s}(z,w))1_A)\rho(z)\rho(w)dzdw = \mathbb{E}((M_s^2 + \mathcal{E}_{H_s}(\rho))1_A).\end{aligned}$$

Hence $(M_t : t \geqslant 0)$ and $(M_t^2 + \mathcal{E}_{H_t}(\rho) : t \geqslant 0)$ are continuous martingales. Thus $(M_t : t \geqslant 0)$ has quadratic variation process $[M]_t = \mathcal{E}_{\mathbb{H}}(\rho) - \mathcal{E}_{H_t}(\rho)$. Set $E_t = \exp\{iM_t - \mathcal{E}_{H_t}(\rho)/2\}$. By Itô's formula, $(E_t : t \geqslant 0)$ is a local martingale, which is moreover bounded. So

$$\mathbb{E}\left(\exp\{iM_t - \mathcal{E}_{H_t}(\rho)/2\}\right) = \mathbb{E}(E_t) = \mathbb{E}(E_0) = \exp\{iM_0 - \mathcal{E}_{\mathbb{H}}(\rho)/2\}. \tag{44}$$

On letting $t \to \infty$, using bounded convergence, we obtain the claimed identity. □

15.4 Schramm–Sheffield theorem

Proposition 15.12 can be interpreted in terms of the characteristic functions of certain Gaussian free fields, and then implies immediately the following result of Schramm and Sheffield, which expresses an identity in law for the corresponding fields. The constant λ appearing in the theorem is affected by our choice of normalization for the Green function so differs by a factor of $\sqrt{2}$ from the original paper.

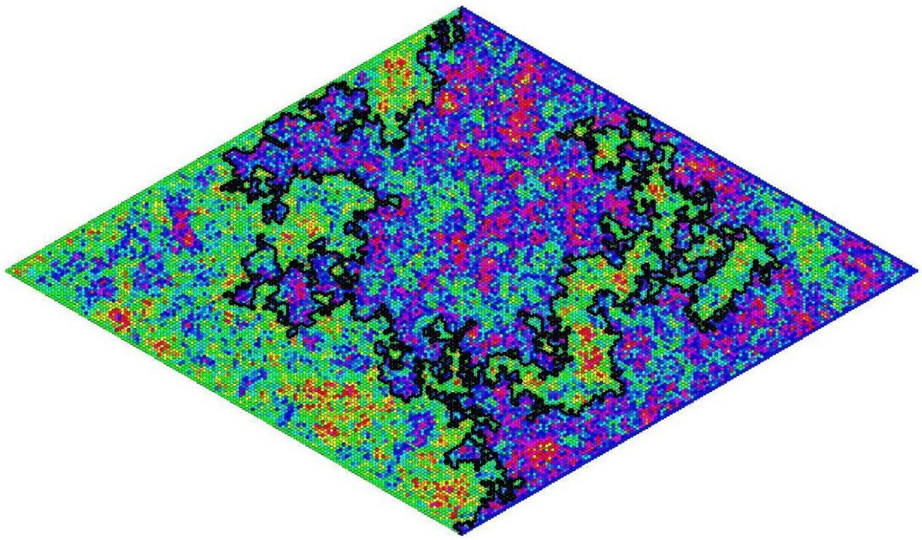


Figure 5: Coupling between Gaussian free field and SLE(4). Picture courtesy of S. Sheffield.

Theorem 15.13. *Let γ be an SLE(4) and let D^- and D^+ be the left and right components of $\mathbb{H} \setminus \gamma^*$. Conditional on γ, let Γ^- and Γ^+ be independent Gaussian free fields with zero boundary values, on D^- and D^+ respectively. Write $\bar{\Gamma}^{\pm}$ for their extensions as random variables in $\mathcal{D}'(\mathbb{H})$. Set $\lambda = \sqrt{\pi/4}$ and define*

$$\Gamma = (\bar{\Gamma}^+ + \lambda 1_{D^+}) - (\bar{\Gamma}^- + \lambda 1_{D^-})$$

Then Γ is a Gaussian free field on $\mathbb{H}$ with boundary values $-\lambda$ and λ on the left and right half-lines respectively.

Before giving the proof, here is a motivating argument, which is not rigorous. *'Suppose we can find a simple chord $\gamma = (\gamma_t : t \geqslant 0)$ in $(\mathbb{H}, 0, \infty)$, parametrized by half-plane capacity, along which there is a cliff in Γ, with value λ to the right and $-\lambda$ to the left. Indeed, suppose we can find γ without looking at the values of Γ away from the cliff. Then, by the Markov property and conformal invariance of the free field, conditional on $\mathcal{F}_t = \sigma(\gamma_s : s \leqslant t)$, $\tilde{g}_t(\Gamma|_{H_t})$ has the original distribution of Γ, and so γ has the domain Markov property. Moreover, by conformal invariance of the free field, γ is also scale invariant, so γ is an SLE(κ) for some $\kappa \in [0, \infty)$. Consider the function $\phi_t(z) = \mathbb{E}(\Gamma(z)|\mathcal{F}_t)$. Then for fixed t, ϕ_t must be the harmonic extension in H_t of the boundary values of Γ on δH_t. Thus $\phi_t = \lambda\sigma_t(z)$. Now $(\phi_t(z) : t < \zeta(z))$ appears to be a martingale. Hence, as we saw in the proof of Proposition 15.11, we must have $\kappa = 4$.'* Note that the theorem turns the construction backwards and does not state that γ is a measurable function of Γ.

Proof of Theorem 15.13. By Proposition 15.12, for all $\rho \in \mathcal{D}(\mathbb{H})$,

$$\mathbb{E}(\exp\{i\Gamma(\rho)\}) = \exp\{i\mathcal{H}_{\mathbb{H}}(\lambda s_0, \rho) - \mathcal{E}_{\mathbb{H}}(\rho)/2\}$$

so $\Gamma(\rho)$ is Gaussian of mean $\mathcal{H}_{\mathbb{H}}(\lambda s_0, \rho)$ and variance $\mathcal{E}_{\mathbb{H}}(\rho)$ by uniqueness of characteristic functions, and so Γ is a Gaussian free field on $\mathbb{H}$ with boundary value λs_0, as required. $\square$

The finite-time identity (44) can be interpreted similarly. *Conditional on $(\gamma_s : s \leqslant t)$, let Γ_t be a Gaussian free field on H_t with boundary value λs_t and let $\bar{\Gamma}_t$ be its extension as a random variable in $\mathcal{D}'(\mathbb{H})$. Then Γ_t is a Gaussian free field on $\mathbb{H}$ with boundary value λs_0.*

16 Appendix

We prove a result of Beurling which concerns the probability that complex Brownian motion $(B_t)_{t\geqslant 0}$ starting from 0 hits a relatively closed subset $A \subseteq \mathbb{D}$ before leaving $\mathbb{D}$. It states that the probability does not increase if we replace A by its *radial projection* $A^* = \{|z| : z \in A\}$. For $A = [\varepsilon, 1)$ we can compute the hitting probability exactly. This provides general source of lower bounds for harmonic measure. We also prove a symmetry estimate, in the case where A is a simple path, for the probability that Brownian motion hits a given side of A. Finally, we prove a maximal inequality for H^1 functions of Brownian motion.

16.1 Beurling's projection theorem

Write T_A for the hitting time of A given by

$$T_A = \inf\{t \geqslant 0 : B_t \in A\}.$$

Theorem 16.1. *Let A be a relatively closed subset of $\mathbb{D}$. Then*

$$\mathbb{P}_0(T_{A^*} < T(\mathbb{D})) \leqslant \mathbb{P}_0(T_A < T(\mathbb{D})).$$

The proof relies on the following folding inequality[14]. Define the folding map ϕ on $\mathbb{C}$ by $\phi(x + iy) = x + i|y|$.

Lemma 16.2. *Let A be a relatively closed subset of $\mathbb{D}$. Then*

$$\mathbb{P}_0(T_{\phi(A)} < T(\mathbb{D})) \leqslant \mathbb{P}_0(T_A < T(\mathbb{D})).$$

Proof. We exclude the case where $0 \in A$ for which the inequality is clear. Consider the set $\rho(A) = A \cup \{\bar{z} : z \in A\}$, symmetrized by reflection. Set $R_0 = 0$ and define, recursively for $k \geqslant 1$,

$$S_k = \inf\{t \geqslant R_{k-1} : B_t \in \rho(A) \text{ or } B_t \notin \mathbb{D}\}, \quad R_k = \inf\{t \geqslant S_k : B_t \in \mathbb{R}\}.$$

Then S_k and R_k are stopping times and $R_{k-1} \leqslant S_k \leqslant R_k < \infty$ for all k, almost surely.

Set $K = \inf\{k \geqslant 1 : S_k = R_k\}$, where we take $\inf \emptyset = \infty$ as usual. On the event $\{K = \infty\}$, we have $R_{k-1} < S_k < R_k < T_{\mathbb{R}\setminus\mathbb{D}} < \infty$ for all k, so the sequences $(S_k : k \geqslant 1)$ and $(R_k : k \geqslant 1)$ have a common accumulation point $T^* \leqslant T_{\mathbb{R}\setminus\mathbb{D}}$. We can write $(\mathbb{D}\setminus\rho(A))\cap\mathbb{R}$ as a countable union of disjoint open intervals $\cup_n I_n$. By a straightforward harmonic measure estimate, there is a constant $C < \infty$ such that $\mathbb{P}_0(B_t \in I_n \text{ for some } t < T_{\mathbb{R}\setminus\mathbb{D}}) \leqslant C\,\mathrm{Leb}(I_n)$ for all n so, by Borel-Cantelli, almost surely, B visits only finitely many of the intervals I_n before $T_{\mathbb{R}\setminus\mathbb{D}}$. Hence, on $\{K = \infty\}$, almost surely $B_{T^*} = \lim_k B_{S_k} = \lim_k B_{R_k}$ is an endpoint

[14]Our proof of the folding inequality is new, though based on ideas from an argument of Oksendal. Whereas Oksendal cuts up the events whose probabilities are to be compared into pieces where symmetry can be invoked to make the comparison, we obtain the inequality from a global inclusion of events, using stochastic calculus to obtain the needed symmetry.

of one of the intervals I_n. But, almost surely, B does not hit any of these endpoints. Hence $K < \infty$ almost surely.

Set $A^+ = \phi(A) \cap A$ and $A^- = \phi(A) \setminus A$. Take a sequence of independent random variables $(\varepsilon_k)_{k \geqslant 1}$, independent of B, with $\mathbb{P}(\varepsilon_k = \pm 1) = 1/2$ for all k. Set $\hat{\varepsilon}_k = \delta_k \varepsilon_k$, where $\delta_k = \pm 1$ according as $B_{S_k} \in \rho(A^\pm)$. Then $(\hat{\varepsilon}_k)_{k \geqslant 1}$ has the same distribution as $(\varepsilon_k)_{k \geqslant 1}$ and is also independent of B. Write $B_t = X_t + iY_t$ and define new processes $\tilde{B}$, $\tilde{Y}$ and $\hat{B}$ by setting

$$\tilde{B}_t = X_t + i\tilde{Y}_t = X_t + i\varepsilon_k Y_t, \quad \hat{B}_t = X_t + i\hat{\varepsilon}_k Y_t, \quad \text{for} \quad R_{k-1} \leqslant t < R_k, \quad k = 1, \dots K$$

and $\tilde{B}_t = X_t + i\tilde{Y}_t = \hat{B}_t = B_t$ for $t \geqslant R_K$. Then we have

$$\tilde{Y}_t = \int_0^t \left(\sum_{k=1}^K \varepsilon_k 1_{\{R_{k-1} \leqslant s < R_k\}} + 1_{\{s \geqslant R_K\}} \right) dY_s$$

almost surely, where the right hand side is understood as an Itô integral in the filtration $(\mathcal{F}_t)_{t \geqslant 0}$ given by

$$\mathcal{F}_t = \sigma(\varepsilon_k, B_s : k \geqslant 1, s \leqslant t).$$

Thus $\tilde{Y}$ is a continuous $(\mathcal{F}_t)_{t \geqslant 0}$-local martingale with quadratic variation $[\tilde{Y}]_t = t$. Morever we have $[X, \tilde{Y}] = 0$. Hence $\tilde{B}$ is a Brownian motion by Lévy's characterization, Similarly $\hat{B}$ is also a Brownian motion (in a different filtration). Note that, with obvious notation, for all k,

$$\tilde{T}(\mathbb{D}) = \hat{T}(\mathbb{D}) = T(\mathbb{D}), \quad \tilde{S}_k = S_k, \quad \tilde{R}_k = R_k.$$

Suppose that $\tilde{B}$ hits $\phi(A)$ before $T(\mathbb{D})$. Note that $\tilde{B}$ cannot hit $\phi(A)$ before S_1 and, if it does not hit $\phi(A)$ at S_k, then it cannot do so until S_{k+1}. Also, if $R_K < T(\mathbb{D})$, then $\tilde{B}_{R_K} \in \rho(A) \cap \mathbb{R} \subseteq \phi(A)$. Hence the only possible values for $T_{\phi(A)}$ are $S_1, \dots, S_K$ and R_K. Now, if $T_{\phi(A)} = S_k$ for some $k \leqslant K$, then either $\tilde{B}_{S_k} \in A^+$ so $\hat{B}_{S_k} = \tilde{B}_{S_k} \in A$, or $\tilde{B}_{S_k} \in A^-$ so $\hat{B}_{S_k} = \bar{\tilde{B}}_{S_k} \in A$. On the other hand, if $T_{\phi(A)} = R_K$, then $\hat{B}_{R_K} = \tilde{B}_{R_K} \in \rho(A) \cap \mathbb{R} \subseteq A$. In all cases $\hat{B}$ hits A before $T(\mathbb{D})$. Hence $\{\tilde{T}_{\phi(A)} < T(\mathbb{D})\} \subseteq \{\hat{T}_A < T(\mathbb{D})\}$ and the folding inequality follows on taking probabilities. $\square$

Proof of Theorem 16.1. The map ϕ folds $\mathbb{C}$ along $\mathbb{R}$ and fixes the point i. Note that ϕ preserves the class of relatively closed subsets of $\mathbb{D}$. Set $\phi_0 = \phi$ and consider for $n \geqslant 1$ the map ϕ_n which folds $\mathbb{C}$ along $\exp(2^{-n}\pi i)\mathbb{R}$ and fixes 1. Set $\psi_n = \phi_n \circ \dots \circ \phi_0$. For all $n \geqslant 0$, by the folding inequality and rotation invariance,

$$\mathbb{P}_0(T_{\phi_n(A)} < T(\mathbb{D})) \leqslant \mathbb{P}_0(T_A < T(\mathbb{D}))$$

and so by induction

$$\mathbb{P}_0(T_{\psi_n(A)} < T(\mathbb{D})) \leqslant \mathbb{P}_0(T_A < T(\mathbb{D})).$$

Consider the set

$$A(n) = \{ze^{i\theta} : z \in A, \, |\theta| \leqslant 2^{-n}\pi\}.$$

Then $A(n)$ is relatively closed and

$$A^* = A(n)^* \subseteq \{xe^{i\theta} : x \in A^*,\ \theta \in [0, 2^{-n}\pi]\} = \psi_n(A(n))$$

so

$$\mathbb{P}_0(T_{A^*} < T(\mathbb{D})) \leqslant \mathbb{P}_0(T_{\psi_n(A(n))} < T(\mathbb{D})) \leqslant \mathbb{P}_0(T_{A(n)} < T(\mathbb{D})).$$

On letting $n \to \infty$ we have $T_{A(n)} \uparrow T_A$ almost surely, so we obtain

$$\mathbb{P}_0(T_{A^*} < T(\mathbb{D})) \leqslant \mathbb{P}_0(T_A \leqslant T(\mathbb{D})).$$

By scaling, the same inequality holds when $\mathbb{D}$ is replaced by $s\mathbb{D}$ for any $s \in (0, 1)$ and the result follows on taking the limit $s \to 1$. □

Theorem 16.3 (Beurling's estimate)**.** *Let A be a relatively closed subset of $\mathbb{D}$ and let $\varepsilon \in (0, 1)$. Suppose that A contains a continuous path from the circle $\{|z| = \varepsilon\}$ to the boundary $\partial\mathbb{D}$. Then*

$$\mathbb{P}_0(T_A \geqslant T(\mathbb{D})) \leqslant 2\sqrt{\varepsilon}.$$

Proof. By the intermediate value theorem, we must have $[\varepsilon, 1) \subseteq A^*$. Then, by Beurling's projection theorem, it will suffice to consider the case where $A = [\varepsilon, 1)$. Consider the conformal map $\mathbb{D} \setminus A \to \mathbb{H}$ given by $\phi = \phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1$, where

$$\phi_1(z) = i\frac{1-z}{1+z}, \quad \phi_2(z) = \frac{1+\varepsilon}{1-\varepsilon}z, \quad \phi_3(z) = \sqrt{z^2+1}, \quad \phi_4(z) = az, \quad a = \frac{1-\varepsilon}{2\sqrt{\varepsilon}}.$$

Then $\phi(0) = i$ and the left and right sides of A are mapped to the interval $(-a, a)$. Then, by conformal invariance of Brownian motion,

$$\mathbb{P}_0(T_A \geqslant T(\mathbb{D})) = \mathbb{P}_i(|B_{T(\mathbb{H})}| > a) = \frac{2}{\pi}\cot^{-1} a$$

and the claimed estimate follows using the bound $\sin x \geqslant 2x/\pi$ for $x \in [0, \pi/2]$. □

16.2 A symmetry estimate

The following symmetry estimate was used in the proof of the restriction property for SLE(8/3).

Proposition 16.4. *Let $\gamma : \mathbb{R} \to \mathbb{C}$ be a simple curve, differentiable at 0 with $\gamma(0) = 0$ and $\dot\gamma(0) \neq 0$. Set $A = \gamma((-\infty, 0])$ and $D = \mathbb{D} \setminus A$, and assume that D is simply connected. Write $A^\pm$ for the left and right sides of $A \cap \mathbb{D}$ in δD. Then (using the notation $\hat{B}_{T(D)}$ from Section 3.1)*

$$\lim_{t\downarrow 0} \mathbb{P}_{\gamma_t}(\hat{B}_{T(D)} \in A^+) = \lim_{t\downarrow 0} \mathbb{P}_{\gamma_t}(\hat{B}_{T(D)} \in A^-) = 1/2.$$

Proof. By rotation invariance, it will suffice to consider the case where $\dot{\gamma}(0) \in (0, \infty)$. For $r \in (0, 1]$, set $\tau(r) = \inf\{t \geqslant 0 : |\gamma(-t)| = r\}$. We deduce from the hypothesis that D is simply connected that $\tau(1) < \infty$ and $|\gamma(-t)| \geqslant 1$ for all $t \geqslant \tau(1)$. Given $\varepsilon > 0$, there exists $r_0 \in (0, 1]$ such that, for all $t \in (0, \tau(r_0))$, we have $|\arg \gamma(t)| \leqslant \varepsilon$ and $|\arg \gamma(-t) - \pi| \leqslant \varepsilon$. Then there exists $r \in (0, r_0)$ such that $|\gamma(-t)| \geqslant r$ for all $t \in [\tau(r_0), \tau(1))$. Define $A(r) = \gamma((-\tau(r), 0])$ and $D(r) = (r\mathbb{D}) \setminus A(r)$. Then $D(r)$ is simply connected. Write $A^+(r)$ for the right side of $A(r)$ in $\delta D(r)$. Then, for $t \in (0, r)$,

$$\mathbb{P}_{\gamma_t}(\hat{B}_{T(D)} \in A^+) \geqslant \mathbb{P}_{\gamma_t}(\hat{B}_{T(D(r))} \in A^+(r))$$

and

$$\liminf_{t \downarrow 0} \mathbb{P}_{\gamma_t}(\hat{B}_{T(D(r))} \in A^+(r)) \geqslant \mathbb{P}_{e^{-2i\varepsilon}}(B \text{ hits } (-\infty, 0] \text{ from above}) = \frac{1}{2} - \frac{\varepsilon}{\pi}$$

where we used a scaling argument for the inequality and the fact that $\arg(B)$ is a local martingale for the equality. By symmetry, and since $\varepsilon > 0$ was arbitrary, this proves the result. $\square$

16.3 A Dirichlet space estimate for Brownian motion

Let B be a Brownian motion in $\mathbb{R}^d$ with B_0 uniformly distributed on $(0, 1]^d$. The projection W of B on the torus $\mathbb{T} = \mathbb{R}^d/\mathbb{Z}^d$ is then a Brownian motion in $\mathbb{T}$, which has the property of reversibility, that is, for all $T > 0$, if we set $\hat{W}_t = W_{T-t}$, then $(W_t)_{0 \leqslant t \leqslant T}$ and $(\hat{W}_t)_{0 \leqslant t \leqslant T}$ have the same distribution on $C([0, T], \mathbb{T})$.

Lemma 16.5. *Let $f \in L^2(\mathbb{T}) \cap H^1(\mathbb{T})$. Then there exists a continuous random process X such that, for all $t \geqslant 0$, we have $X_t = f(W_t)$ almost surely and*

$$\left\| \sup_{s \leqslant t} |X_s| \right\|_2 \leqslant \|f\|_{L^2(\mathbb{T})} + (5/\sqrt{2})\sqrt{t}\|f\|_{H^1(\mathbb{T})}.$$

Proof. Consider first the case where $f \in C^2(\mathbb{T})$. Fix $T > 0$ and write $\hat{W}$ for the time-reversal from T as above. By Itô's formula,

$$f(W_t) = f(W_0) + M_t + \frac{1}{2}\int_0^t \Delta f(W_s)ds, \quad M_t = \int_0^t \nabla f(W_s)dW_s.$$

On the other hand, Itô's formula may be applied also to $\hat{W}$ to give

$$f(W_t) = f(\hat{W}_{T-t}) = f(\hat{W}_0) + \hat{M}_{T-t} + \frac{1}{2}\int_0^{T-t} \Delta f(\hat{W}_s)ds, \quad \hat{M}_t = \int_0^t \nabla f(\hat{W}_s)d\hat{W}_s.$$

We subtract the corresponding formula for $t = 0$ to obtain

$$f(W_t) = f(W_0) + \hat{M}_{T-t} - \hat{M}_T - \frac{1}{2}\int_0^t \Delta f(W_s)ds.$$

Then, by adding, we obtain the Lyons–Zheng decomposition

$$f(W_t) = f(W_0) + \frac{1}{2}(M_t + \hat{M}_{T-t} - \hat{M}_T)$$

and hence the inequality

$$\sup_{t \leqslant T} |f(W_t)| \leqslant |f(W_0)| + \frac{1}{2}\left(\sup_{t\leqslant T}|M_t| + \sup_{t\leqslant T}|\hat{M}_t| + |\hat{M}_T|\right).$$

Now

$$\mathbb{E}(|f(W_0)|^2) = \int_{\mathbb{T}} |f(x)|^2 dx = \|f\|^2_{L^2(\mathbb{T})}$$

and, by Doob's L^2-inequality, the Itô isometry and Fubini,

$$\mathbb{E}\left(\sup_{t\leqslant T}|M_t|^2\right) \leqslant 4\mathbb{E}(|M_T|^2) = 4\mathbb{E}\int_0^T |\nabla f(W_t)|^2 dt = 4T\int_{\mathbb{T}} |\nabla f(x)|^2 dx = 8T\|f\|^2_{H^1(\mathbb{T})}.$$

The same holds for $\hat{M}$. Hence we obtain

$$\left\|\sup_{t\leqslant T}|f(W_t)|\right\|_2 \leqslant \|f\|_{L^2(\mathbb{T})} + (5/\sqrt{2})\sqrt{T}\|f\|_{H^1(\mathbb{T})}.$$

We return to the general case, where $f \in L^2(\mathbb{T}) \cap H^1(\mathbb{T})$. There exist functions $f_n \in C^2(\mathbb{T})$ such that $\|f_n - f\|_{L^2(\mathbb{T})} + \|f_n - f\|_{H^1(\mathbb{T})} \leqslant 2^{-n}$ for all n. Set $X^n_t = f_n(W_t)$. The estimate just obtained applies to the functions $f_n - f_{n+1}$ to show that

$$\sum_{n=1}^{\infty} \left\|\sup_{t\leqslant T}|X^n_t - X^{n+1}_t|\right\|_2 < \infty.$$

Hence, almost surely, and uniformly in $t \leqslant T$ for all $T \geqslant 0$, the sequence (X^n_t) is Cauchy, and hence convergent, with continuous limit $(X_t)_{t\geqslant 0}$ say. Now

$$\mathbb{E}(|f_n(W_t) - f(W_t)|^2) = \|f_n - f\|^2_{L^2(\mathbb{T})}, \quad \left\|\sup_{t\leqslant T}|f_n(W_t)|\right\|_2 \leqslant \|f_n\|_{L^2(\mathbb{T})} + (5/\sqrt{2})\sqrt{T}\|f_n\|_{H^1(\mathbb{T})}.$$

On letting $n \to \infty$, by Fatou's lemma, we obtain $\mathbb{E}(|X_t - f(W_t)|^2) = 0$, so $X_t = f(W_t)$ almost surely, and we also obtain the desired estimate for $\|\sup_{t\leqslant T}|X_t|\|_2$. $\square$

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