Introduction to Schramm–Loewner evolutions

J.R. Norris

February 17, 2010

1 Preliminaries and overview

Every simply connected planar domain, except $\emptyset$ and $\mathbb{C}$, is the image of the unit disc $\mathbb{D}$ under some conformal isomorphism[1]. We shall write $\hat{U}$ for the completion of such a domain U in the metric thereby inherited from the Euclidean metric on the unit disc $\mathbb{D}$. The metric space $\hat{U}$ depends on the choice of isomorphism only up to a uniformly equivalent change of metric. Given a triple $D = (U, z_0, z_1)$, with $z_0, z_1 \in \hat{U}$, by a *path* in D we mean a continuous function $[0,1] \to \hat{U}$, starting from z_0 and ending at z_1.

We call any increasing homeomorphism on $[0,1]$ a *reparametrization*. By a *chord* in D we mean an equivalence class of paths modulo reparametrizations. Write C_D for the set of chords in D. Given a path γ, we write $[\gamma]$ for the associated chord; more generally, if γ is a continuous map from any compact subinterval I of $[0,\infty]$ to $\hat{U}$, starting at z_0 and ending at z_1, we can obtain a chord $[\gamma]$ in D by choosing an increasing homeomorphism $\phi : [0,1] \to I$ and setting $[\gamma] = [\gamma \circ \phi]$.

Write $\mathcal{D}$ for the set of triples $D = (U, z_0, z_1)$, where U is a simply connected proper (that is, not $\emptyset$ or $\mathbb{C}$) planar domain and where z_0, z_1 are distinct points in $\hat{U} \setminus U$. For $D \in \mathcal{D}$, by a *filling* in D we mean a closed, connected, simply connected subset of $\hat{U}$ containing z_0 and z_1. Write S_D for the set of fillings in D.

We shall study some families of probability measures $(\mu_D : D \in \mathcal{D})$ where, for each D, μ_D is the distribution of a random chord or a random filling in D. We defer until Sections 7 and 9 the specification of σ-algebras on C_D and S_D needed to make these notions precise.

For $D, D' \in \mathcal{D}$, a *conformal isomorphism* $\Phi : D \to D'$ is a conformal isomorphism $U \to U'$ with $\Phi(z_0) = z_0'$ and $\Phi(z_1) = z_1'$. The set of conformal automorphisms of D forms a one-parameter group, whose elements we call *scalings*, by analogy with the case of the upper half-plane $\mathbb{H}$ with boundary points 0 and ∞, where the conformal automorphisms are given by $\Phi_\lambda(z) = \lambda z$ for $\lambda \in (0,\infty)$. We say that the measure μ_D is *scale-invariant* if, whenever $X \sim \mu_D$, then also $\Phi(X) \sim \mu_D$, for any scaling Φ.

Example 1.1. Let $(R_t)_{t\geqslant 0}$ be a three-dimensional Bessel process (which can be realised as the modulus of a Brownian motion in $\mathbb{R}^3$) and let $(B_t)_{t\geqslant 0}$ be a Brownian motion, both starting from 0. Set $E_t = B_t + iR_t$. We know that $R_t \to \infty$ as $t \to \infty$, so E extends

---

[1] In this introductory section, in an attempt to convey briefly what are stochastic Loewner evolutions and why they are of interest, we assume some prior knowledge of complex analysis and stochastic analysis. A reader who is content to wait for motivation and orientation until later may prefer to start at Section 2.

continuously to $I = [0, \infty]$ as a map into $\hat{\mathbb{H}}$. Then $X = [E]$ is a random chord in $(\mathbb{H}, 0, \infty)$. The usual Brownian scaling property implies that the associated law is scale-invariant.

Given a scale-invariant measure μ_{D_0} for one $D_0 \in \mathcal{D}$, there is a natural way to obtain a whole family $(\mu_D : D \in \mathcal{D})$: take μ_D to be the law of $\Phi(X)$, where $X \sim \mu_{D_0}$ and $\Phi : D_0 \to D$ is a conformal isomorphism. Note that μ_D is scale-invariant and does not depend on the choice of Φ, because of the scale invariance of μ_{D_0}. Moreover the family $(\mu_D : D \in \mathcal{D})$ has the property of *conformal invariance*: for all $D, D' \in \mathcal{D}$ and any conformal isomorphism $\Phi : D \to D'$, if $X \sim \mu_D$, then $\Phi(X) \sim \mu_{D'}$.

Example 1.2 (Brownian excursion). We refer to the preceding example and obtain, by the procedure just described, from the scale-invariant law of X on chords in $(\mathbb{H}, 0, \infty)$, a conformally invariant family $(\mu_D : D \in \mathcal{D})$. For each $D \in \mathcal{D}$, the measure μ_D may also be obtained as follows. Choose $w_0 \in D$ and a neighbourhood N of z_1 in $\hat{U}$. Let $Z = (Z_t)_{t \leqslant T}$ be a complex Brownian motion, starting from w_0 and run up to the first time T that it exits $U \setminus \bar{N}$. We know that $T < \infty$ almost surely and that $\mathbb{P}(Z_T \in \bar{N}) > 0$. Condition on $Z_T \in \bar{N}$ and set $X = [Z]$. Then the law of X converges weakly as $w_0 \to z_0$ and $N \downarrow z_1$, with limit μ_D.

We shall study the conjunction of conformal invariance with some other properties, each of which derives from a natural relationship between domains.

1.1 Restriction property

For $D = (U, z_0, z_1)$ and $D' = (U', z'_0, z'_1)$ in $\mathcal{D}$, write $D' \subseteq D$ if $U' \subseteq U$, with $U' = U$ near z_0 and z_1, and $z'_0 = z_0$, $z'_1 = z_1$. Let $(\mu_D : D \in \mathcal{D})$ be a family of probability measures on fillings. Say that $(\mu_D : D \in \mathcal{D})$ has the *restriction property* if, for all $D, D' \in \mathcal{D}$ with $D' \subseteq D$, the conditional law of a random filling $X \sim \mu_D$, given $X \subseteq D'$, is $\mu_{D'}$.

Example 1.3 (Self-avoiding walk). This and the following two examples are given to illustrate how the properties we define appear naturally in some discrete settings. We do not attempt a wholly precise account, leaving aside, in particular, some technical questions concerning how to adapt the continuum domain D to a discrete lattice. We shall not pursue these examples in the sequel. Consider, for $\varepsilon > 0$ and $\alpha \in (0, 1)$, the probability measure μ_D^ε on self-avoiding lattice paths in $D \cap \varepsilon\mathbb{Z}^2$ joining z_0 and z_1, which gives a weight proportional to α^{-n} to paths of length n. Then the family $(\mu_D^\varepsilon : D \in \mathcal{D})$ has the restriction property. It is conjectured that, for a certain critical value of α, independent of D, the limit as $\varepsilon \to 0$ of these measures exists and is conformally invariant.

1.2 Locality property

Let $D = (U, z_0, z_1)$ in $\mathcal{D}$. On removing z_0 and z_1 from $\hat{U} \setminus U$, we obtain two connected components. A simple path $c : [0, 1] \to \hat{U}$, starting from one component and ending in the other, and taking $(0, 1)$ into U, will be called a *cut* of D. On removing a such a cut from U, we obtain two subdomains U_0 and U_1 with $z_0 \in \hat{U}_0$ and $z_1 \in \hat{U}_1$. We call the pair $N = (U_0, z_0)$ an *initial domain* in D. Let $(\mu_D : D \in \mathcal{D})$ be a family of probability measures on chords. Say that $(\mu_D : D \in \mathcal{D})$ has the *locality property* if, for all $D, D' \in \mathcal{D}$, for all initial domains N common to D and D', for $X \sim \mu_D$ and $X' \sim \mu_{D'}$, we have $X^N \sim X'^N$, where X^N is X stopped on hitting the cut.

Example 1.4 (Percolation). Consider, for $\varepsilon > 0$, a honeycomb lattice with edge $[-i\varepsilon, i\varepsilon]$. Colour each cell independently black or white with probability 1/2. Given $D \in \mathcal{D}$, turn every cell in ∂D^+ white and every cell in ∂D^- black. Write X_D^ε for the unique chord in D with white cells on the right and black cells on the left. If N is an initial domain for both D and D', then $X_D^{\varepsilon,N} = X_{D'}^{\varepsilon,N}$, so the associated family of laws $(\mu_D^\varepsilon : D \in \mathcal{D})$ has the locality property. A result of Smirnov states that the limit as $\varepsilon \to 0$ of these measures exists and is conformally invariant.

1.3 Domain Markov property

Let us say that a path $\gamma \in P_D$ is regular if it has no interval of constancy and if $\gamma_t \neq z_1$ for all $t < 1$. For such a path, for all $t < 1$, the open set $U \setminus \gamma[0,t]$ has a unique connected component which is a neighbourhood of z_1. We denote this component by U_t. The inclusion $U_t \to U$ extends to a continuous map $i_t : \hat{U}_t \to \hat{U}$. See [2, pp.88–89] and [3, Theorem 2.1] for details. We say that γ is *non-self-traversing* if it is regular and, for all $t < 1$, there exist unique points $\tilde{z}_0, \tilde{z}_1 \in \hat{U}_t$ and a unique regular path $(\tilde{\gamma}_s)_{t \leqslant s \leqslant 1}$ from $\tilde{z}_0$ to $\tilde{z}_1$ in U_t such that $i_t(\tilde{\gamma}_s) = \gamma_s$ for all $t \leqslant s \leqslant 1$. We write $\tilde{\gamma} = \theta_t \gamma$ and allow ourselves to write γ_t for $\tilde{z}_0$ and z_1 for $\tilde{z}_1$. This property is invariant under reparametrization, so it can be applied to chords.

A function $\tau : P_D \to [0,1]$ is *parametrization-invariant* if $\tau(\gamma \circ \phi) = \phi^{-1}(\tau(\gamma))$ for all $\gamma \in P_D$ and all reparametrizations ϕ. For such τ and for a non-self-traversing chord $x = [\gamma] \in C_D$, we can define $x_\tau = \gamma_{\tau(\gamma)}$ and also a *stopped chord* $x^\tau = [\gamma|_{[0,\tau(\gamma)]}]$ in (U, z_0, x_τ) and a *shifted chord* $\theta_\tau x$ in (U_τ, x_τ, z_1).

Let $(\mu_D : D \in \mathcal{D})$ be a family of probability measures on non-self-traversing chords. We say that $(\mu_D : D \in \mathcal{D})$ has the *domain Markov property* if, for all $D = (U, z_0, z_1) \in \mathcal{D}$, for $X \sim \mu_D$, X does not hit z_1 before time 1 almost surely, and, for all parametrization-invariant stopping times τ on P_D, conditional on the stopped chord X^τ and on $\tau(X) < 1$, we have $\theta_\tau X \sim \mu_{D_\tau}$, where $D_\tau = (U_\tau(X), X_\tau, z_1)$.

Example 1.5 (Loop-erased random walk). Consider, for $\varepsilon > 0$, a simple random walk S in $\varepsilon\mathbb{Z}^2$, starting from z_0 and run until it leaves U, at time T say. Condition on $S_T = z_1$ and then erase loops from S as follows: set $T_0 = 0$ and $\tilde{S}_0 = S_0$, and define recursively, for $n \geqslant 0$ until $\tilde{S}_n = z_1$,

$$T_{n+1} = \sup\{m \leqslant T : S_m = S_{T_n+1}\}, \quad \tilde{S}_{n+1} = S_{T_{n+1}}.$$

Thus we obtain a simple random chord in D. By an elementary argument, it can be shown that the associated family of laws $(\mu_D^\varepsilon : D \in \mathcal{D})$ has (a suitable discrete version of) the domain Markov property. It was the consideration of possible continuum limits for this example which led Schramm in 1999 to the discovery of SLE. The convergence of the measures as $\varepsilon \to 0$ to a conformally invariant limit has now been shown by Lawler, Schramm and Werner.

2 Brownian motion and conformal isomorphisms

A real-valued random process $B = (B_t)_{t \geqslant 0}$, with continuous paths and starting from 0, is a *Brownian motion* if B has stationary and independent increments and $B_t \sim N(0,t)$ for all t. On displacing the whole process by $x \in \mathbb{R}$ we obtain a Brownian motion *starting from* x. A *complex Brownian motion starting from* $z = x + iy$ is a complex-valued process whose real and imaginary parts are independent Brownian motions starting from x and y respectively.

A *domain* in $\mathbb{C}$ or $\mathbb{R}^n$ is a connected open set. An *analytic* function f defined on a complex domain D is one which is complex differentiable. For any analytic function $f = u + iv$, its real and imaginary parts are harmonic in D and their partial derivatives satisfy the Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$. An analytic function f on D is a *conformal map* if $f'(z) \neq 0$ for all $z \in D$. A bijective conformal map $\Phi : D \to D'$ has an inverse function $\Phi^{-1} : D' \to D$ which also a conformal map. Thus we shall call such Φ a *conformal isomorphism* from D to D'.

We shall make extensive use of Itô's formula: if D is a domain in $\mathbb{R}^n$ and $u : D \to \mathbb{R}$ is a C^2 function, and if Z is a continuous semimartingale with values in D, then

$$u(Z_t) = u(Z_0) + \int_0^t \partial_i u(Z_s) dZ_s^i + \frac{1}{2} \int_0^t \partial_i \partial_j u(Z_s) dZ_s^i dZ_s^j, \quad t \geqslant 0.$$

Here, the second term on the right is Itô's stochastic integral, and the third term is a Lebesgue-Stieltjes integral with respect to the covariation process $[Z^i, Z^j]$. We sum over the repeated indices i and j. The product differential $dZ_s^i dZ_s^j$ is an alternative notation for $d[Z^i, Z^j]_s$.

There is a version of Itô's formula for complex domains and analytic functions which can be written, in differential form,

$$d(f(Z_t)) = f'(Z_t) dZ_t + (1/2) f''(Z_t) dZ_t dZ_t.$$

Here we are using complex rather than tensor multiplication. It is an exercise to check that, if we write $f = u + iv$ and $Z = X + iY$, multiply out and collect real and imaginary parts, then we obtain the preceding formula for $u(Z_t)$ along with its analogue for $v(Z_t)$.

Note that, if $Z = X + iY$ is a complex Brownian motion, then $dXdX = dYdY = dt$ and $dXdY = 0$, so

$$dZdZ = (dX + idY)(dX + idY) = dXdX - dYdY + 2idXdY = 0.$$

Hence, up to the first time T_D that Z leaves D,

$$f(Z_t) = f(Z_0) + \int_0^t f'(Z_s) dZ_s,$$

which is a (complex) local martingale. Alternatively, considered as a tensor, we have $dZ_t^i dZ_t^j = \delta^{ij} dt$, so $\partial_{ij}^2 u(Z_t) dZ_t^i dZ_t^j = \Delta u(Z_t) dt = 0$ and similarly for v, which gives the same conclusion.

Note that, if M is a local martingale and T is a stopping time, such that $(M_t : t \leqslant T)$ is uniformly bounded, then $\mathbb{E}(M_T) = \mathbb{E}(M_0)$ by optional stopping. In particular, if f is continuous and bounded on $\bar{D}$ and is analytic on D, and if Z is a complex Brownian motion starting from z, then $f(z) = \mathbb{E}(f(Z_{T_D}))$.

Lemma 2.1 (Schwarz lemma)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function with $f(0) = 0$. Then $|f(z)| \leqslant |z|$ for all z. Moreover, if $|f(z)| = |z|$ for some $z \neq 0$, then $f(w) = e^{i\theta} w$ for all w, for some $\theta \in \mathbb{R}$.*

Proof. Fix $z \in \mathbb{D}$ and consider a Brownian motion B starting from z. Fix $r \in (|z|, 1)$ and $\varepsilon \in (0, 1 - |z|)$ and consider the stopping times

$$S = \inf\{t \geqslant 0 : |B_t| = r\}, \quad T = \inf\{t \geqslant 0 : |B_t - z| = \varepsilon\}.$$

Consider the function $g(z) = f(z)/z$. Then g is analytic in $\mathbb{D}$ so, by optional stopping,

$$g(z) = \mathbb{E}(g(B_S)) = \mathbb{E}(g(B_T)).$$

Now $|g(B_S)| \leqslant 1/r$, so, letting $r \uparrow 1$, we deduce that $|g(z)| \leqslant 1$. On the other hand, if $e^{i\theta} g(z) = 1$, then, since $|e^{i\theta} g(w)| \leqslant 1$ and B_T is uniform on $\{w : |w - z| = \varepsilon\}$, we must have $e^{i\theta} g = 1$ near z, and hence everywhere by connectedness. □

Corollary 2.2. *Let $\Phi : \mathbb{D} \to \mathbb{D}$ be a conformal isomorphism. Then Φ is a Möbius transformation: there exists $w \in \mathbb{D}$ and $\theta \in \mathbb{R}$ such that $\Phi(z) = e^{i\theta}(z - w)/(1 - \bar{w}z)$ for all z.*

Proof. Set $\Psi(z) = (z - w)/(1 - \bar{w}z)$, where $w = \Phi^{-1}(0)$. Then $f = \Psi \circ \Phi^{-1}$ is a conformal isomorphism from $\mathbb{D}$ to itself with $f(0) = 0$. By the Schwarz lemma, applied to f or f^{-1}, there exists $\theta \in \mathbb{R}$ such that $f(z) = e^{-i\theta} z$ for all z. □

We shall need the following fundamental result of complex analysis, whose proof we omit.

Theorem 2.3 (Riemann mapping theorem)**.** *Let D be a simply connected proper domain. Then there exists a conformal isomorphism $\Phi : D \to \mathbb{D}$.*

If Φ and Ψ are both conformal isomorphisms from D to $\mathbb{D}$, then $\Phi \circ \Psi^{-1}$ has the form given in Corollary 2.2. So, the Euclidean metric on $\mathbb{D}$ and its image under $\Phi \circ \Psi^{-1}$ are uniformly equivalent. Let us choose one such isomorphism Φ and use it to transfer the metric on $\mathbb{D}$ to D. Form a topological space $\hat{D}$ by completing D in this metric. Then the *conformal completion* $\hat{D}$ and the *conformal boundary* $\hat{D} \setminus D$ do not depend on our choice of Φ. Moreover, Φ extends uniquely to a homeomorphism $\hat{D} \to \bar{\mathbb{D}}$, and indeed any conformal isomorphism extends uniquely to a homeomorphism of the conformal completions. For a conformal boundary point b and a connected open subset N of D, we say that N is a *neighbourhood of b in D* if there exists an $\varepsilon > 0$ such that $\{|z - \Phi(b)| < \varepsilon\} \cap \Delta \subseteq \Phi(N)$. Note that $\hat{\mathbb{H}}$ is the one point compactification $\bar{\mathbb{H}} \cup \{\infty\}$ of $\bar{\mathbb{H}}$.

Let $z \in D$ and let b_1, b_2, b_3 be distinct and positively ordered conformal boundary points of D. By composing with a suitable Möbius transformation[2], we can impose on the map Φ given in the theorem any one of the following additional conditions, each of which serves to specify Φ uniquely:

- $\Phi(z) = 0$ and $\Phi'(z) > 0$;
- $\Phi(z) = 0$ and $\Phi(b_1) = 1$;
- $\Phi(b_1) = 1, \Phi(b_2) = i$ and $\Phi(b_3) = -1$.

This shows that the only conformal automorphisms of $(\mathbb{H}, 0, \infty)$ are the maps $\Phi_\lambda(z) = \lambda z$ and, more generally, for $D \in \mathcal{D}$, there exists a conformal isomorphism $\Psi : D \to (\mathbb{H}, 0, \infty)$ so all conformal automorphisms of D are of the form $\Psi \circ \Phi_\lambda \circ \Psi^{-1}$.

Theorem 2.4 (Conformal invariance of Brownian motion)**.** *Let D and D' be simply connected proper domains, and let $z \in D$ and $z' \in D'$. Let $\Phi : D \to D'$ be a conformal isomorphism taking z to z'. Let B and B' be complex Brownian motions starting from z and z' respectively, and set*

$$T = \inf\{t \geqslant 0 : B_t \notin D\}, \quad T' = \inf\{t \geqslant 0 : B'_t \notin D'\}.$$

Define on $[0, T)$ a random increasing function A by

$$A(t) = \int_0^t |\Phi'(B_s)|^2 ds$$

and write τ for the inverse function $\tau = A^{-1} : [0, A(T)) \to [0, T)$. Set $\tilde{T} = A(T)$ and $\tilde{B}_t = \Phi(B_{\tau(t)})$ for $t < \tilde{T}$. Then $(\tilde{T}, (\tilde{B}_t)_{t<\tilde{T}})$ and $(T', (B'_t)_{t<T'})$ have the same distribution.

Proof. By Itô's formula, as above, the real and imaginary parts M and N of the process $\Phi(B)$, defined up to T, are both local martingales. Moreover, from the stochastic integral representation for M and N we find that both have quadratic variation process A and that $[M, N] = 0$. Hence, by an argument based on Lévy's characterization of Brownian motion, $(\tilde{B}_t)_{t<\tilde{T}}$ is a Brownian motion, and clearly $\tilde{T} = \inf\{t \geqslant 0 : \tilde{B}_t \notin D'\}$. $\square$

Since $T < \infty$ almost surely and, by referring to the case $D = \Delta$ say, we know that B_t converges to a conformal boundary point as $t \uparrow T$, we can define using $(B_t : t < T)$ a random chord X in $\hat{D}$, starting from z and ending at the conformal boundary. A good way to think of the preceding theorem is as saying that the associated family of laws $\mu_{D,z}$ is conformally invariant, combined with the fact that, via its quadratic variation, Brownian motion carries a natural parametrization.

---

[2] For $D = \mathbb{D}$, rotate to put b_1 at 1, use the usual map to $\mathbb{H}$ by $\Psi(z) = -i(z+1)/(z-1)$ so b_1 is now at ∞; the images of b_2 and b_3 can now be placed as desired by an affine map on $\mathbb{H}$, which fixes ∞; then return to $\mathbb{D}$ using Ψ^{-1}. Note that the rotation and affine map are both uniquely determined.

The theorem provides an effective means to calculate the hitting distribution or *harmonic measure* $h_D(z, .)$ of Brownian motion for many domains D and starting points z, by taking $D' = \mathbb{D}$ and $z' = 0$: the harmonic measure on D, starting from z is then the pull back of the uniform distribution on $\partial\mathbb{D}$. Thus, if we parametrize the conformal boundary by t, then the hitting density is given by

$$h_D(z,t) = \frac{1}{2\pi}\frac{d\theta}{dt},$$

where $\theta \in [0, 2\pi]$ is the usual parametrization of $\partial\mathbb{D}$. In particular, for $z \in \mathbb{D}$, we have

$$h_{\mathbb{D}}(z,t) = \frac{1}{2\pi}\frac{1-|z|^2}{|e^{it}-z|^2}, \quad 0 \leqslant t < 2\pi$$

and, for $z = x + iy \in \mathbb{H}$,

$$h_{\mathbb{H}}(z,t) = \frac{1}{\pi}\operatorname{Im}\left(\frac{1}{t-z}\right) = \frac{y}{\pi((x-t)^2+y^2)}, \quad t \in \mathbb{R}.$$

The following lemma allows us to bound the partial derivatives of an analytic function in terms of its supremum norm.

Lemma 2.5. *Let u be a harmonic function in D and let $z \in D$. Then*

$$|u_x(z)| \leqslant \frac{4\|u\|_\infty}{\pi \operatorname{dist}(z, \partial D)}.$$

Proof. By affine transformation, we reduce to the case where $z = 0$ and $\operatorname{dist}(z, \partial D) = 1$. We compute

$$\nabla h_{\mathbb{D}}(0, \theta) = \frac{1}{\pi}(\cos\theta, \sin\theta).$$

For $z \in \mathbb{D}$,

$$u(z) = \int_0^{2\pi} u(e^{i\theta}) h_{\mathbb{D}}(z, \theta) d\theta$$

so

$$\nabla u(0) = \frac{1}{\pi}\int_0^{2\pi} u(e^{i\theta})(\cos\theta, \sin\theta) d\theta$$

and so

$$|u_x(0)| \leqslant \frac{\|u\|_\infty}{\pi}\int_0^{2\pi} |\cos\theta| d\theta = \frac{4\|u\|_\infty}{\pi}.$$

□

3 Half-plane capacity

A bounded subset K of the upper half-plane $\mathbb{H}$ is called a *compact* $\mathbb{H}$*-hull* if $K = \mathbb{H} \cap \bar{K}$ and $\mathbb{H} \setminus K$ is simply connected. We often write $H = \mathbb{H} \setminus K$ and $I = \bar{K} \setminus K$. Note that I is a closed subset of the real axis.

Proposition 3.1. *Let K be a compact $\mathbb{H}$-hull. Then there exists a unique conformal isomorphism $g_K : H \to \mathbb{H}$ such that $g_K(z) - z \to 0$ as $z \to \infty$. Moreover, there is a constant $a_K \in \mathbb{R}$ such that*

$$g_K(z) = z + \frac{a_K}{z} + O(|z|^{-2}), \quad z \to \infty.$$

Furthermore, g_K extends analytically to $\mathbb{R} \setminus I$ with $g_K'(x) \in (0, 1]$ for all $x \in \mathbb{R} \setminus I$.

Proof. Map H to a neighbourhood of 0 in $\mathbb{H}$ by $-1/z$, then reflect in $\mathbb{R}$ to obtain a neighbourhood D of 0 in $\mathbb{C}$. Given $r > 0$, there is a unique conformal isomorphism $\Phi : D \to \mathbb{C} \setminus ((-\infty, -r] \cup [r, \infty))$ with $\Phi(0) = 0$ and $\Phi'(0) > 0$. By scaling, we can choose r uniquely so that $\Phi'(0) = 1$. By reflection symmetry and uniqueness, $\overline{\Phi(z)} = \Phi(\bar{z})$. Hence Φ is real on $D \cap \mathbb{R}$ and, given that $\Phi'(0) = 1$, Φ must restrict to a conformal isomorphism from $D \cap \mathbb{H}$ to $\mathbb{H}$. Then, for some $b, c \in \mathbb{R}$,

$$\Phi(z) = z + bz^2 + cz^3 + O(|z|^4), \quad z \to 0.$$

Define g_K on H by Set $g_K(z) = -1/\Phi(-1/z) - b$, then g_K is a conformal isomorphism to $\mathbb{H}$ and $g_K(z) - z \to 0$ as $z \to \infty$. If $\tilde{g}_K$ is any other such conformal isomorphism, then $f = \tilde{g}_K \circ g_K^{-1}$ is a conformal automorphism of $\mathbb{H}$ with $f(z) - z \to 0$ as $z \to \infty$. But f is a Möbius tranformation by the Schwarz lemma, so we must have $f(z) = z$ for all z, showing that g_K is unique. Moreover g_K extends analytically to $\mathbb{R} \setminus I$ with $g_K'(x) \in (0, \infty)$ for all $x \in \mathbb{R} \setminus I$. It is now a straightforward exercise to check that g_K has the desired expansion at ∞, with $a_K = b^2 - c$.

Let B be a complex Brownian motion starting from $z = x + iy$ and set

$$T = \inf\{t \geqslant 0 : B_t \notin H\}, \quad T_0 = \inf\{t \geqslant 0 : B_t \notin \mathbb{H}\}.$$

Note that, for $a, b \in \mathbb{R}$ with $a < b$, in the limit $y \to \infty$ with $x/y \to 0$, we have

$$\pi y \, \mathbb{P}_z(B_{T_0} \in [a, b]) = \int_a^b \frac{y^2}{y^2 + (t - x)^2} dt \to b - a.$$

Suppose that $[a, b] \cap I = \emptyset$ and $z \in H$, then by conformal invariance of Brownian motion

$$\mathbb{P}_{g_K(z)}(B_{T_0} \in [g_K(a), g_K(b)]) = \mathbb{P}_z(B_T \in [a, b]) \leqslant \mathbb{P}_z(B_{T_0} \in [a, b]).$$

Set $z = iy$ and write $g_K(z) = u + iv$. Note that, as $y \to \infty$, we have $v/y \to 1$, $v \to \infty$ and $u/v \to 0$. Hence on multiplying the preceding inequality by πy and letting $y \to \infty$ we obtain

$$g_K(b) - g_K(a) \leqslant b - a.$$

It follows that $g'(x) \leqslant 1$ for all $x \in \mathbb{R} \setminus I$. □

Proposition 3.2. *Let K be a compact $\mathbb{H}$-hull, with $K \subseteq \mathbb{D}$. Set*

$$a_K(\theta) = \mathbb{E}_{e^{i\theta}}(\operatorname{Im}(B_T)),$$

where B is a complex Brownian motion starting from $e^{i\theta}$ and $T = T(K) = \inf\{t \geqslant 0 : B_t \notin \mathbb{H} \setminus K\}$. Then

$$a_K = \int_0^\pi a_K(\theta) p(\theta) d\theta \geqslant 0 \tag{1}$$

where $p(\theta) = 2\sin\theta/\pi$. Moreover, there is a constant $C < \infty$, independent of K, such that

$$\left| g_K(z) - z - \frac{a_K}{z} \right| \leqslant \frac{C a_K}{|z|^2}, \quad |z| \geqslant 2. \tag{2}$$

Proof. Set $\phi(z) = \operatorname{Im}(z - g_K(z))$, then ϕ is bounded and harmonic on $H = \mathbb{H} \setminus K$. Note that $\operatorname{Im}(g_K(B_t)) \to 0$ as $t \uparrow T$, so, by optional stopping and the strong Markov property,

$$\phi(z) = \mathbb{E}_z(\operatorname{Im}(B_T)) = \int_0^\pi h_D(z, \theta) a_K(\theta) d\theta,$$

where $D = \mathbb{H} \setminus \bar{\mathbb{D}}$. Consider the conformal map $w = \Phi(z) = z + z^{-1} : D \to \mathbb{H}$ and note that $\Phi(e^{i\theta}) = 2\cos\theta$. Then, for $z \in D$,

$$h_D(z, \theta) = h_{\mathbb{H}}(w, 2\cos\theta) \frac{d}{d\theta} \Phi(e^{i\theta}) = \operatorname{Im}\left(\frac{1}{2\cos\theta - w} \right) \frac{2\sin\theta}{\pi}.$$

There is a constant $C < \infty$ such that, for all $|z| \geqslant 3/2$ and $\theta \in (0, \pi)$,

$$\left| \frac{1}{w - 2\cos\theta} - \frac{1}{z} \right| = \frac{|2\cos\theta - z^{-1}|}{|z||z + z^{-1} - 2\cos\theta|} \leqslant \frac{C}{|z|^2}.$$

Set $f(z) = u(z) + iv(z) = g_K(z) - z - a/z$, $z \in H$, where a is given by the integral in (1). Then f is analytic in D and vanishes at ∞. Also

$$v(z) = \operatorname{Im}\left(-\frac{a}{z}\right) - \phi(z) = \int_0^\pi \operatorname{Im}\left(\frac{1}{w - 2\cos\theta} - \frac{1}{z} \right) a_K(\theta) p(\theta) d\theta$$

so $|v(z)| \leqslant Ca/|z|^2$ whenever $|z| \geqslant 3/2$.

Since $v(x) = 0$ for $|x| \geqslant 3/2$, we can extend v by reflection as a harmonic function in $\{|z| > 3/2\}$, with the same bounds. Hence, for $|z| \geqslant 2$, we can apply Lemma 2.5 in the domain $\{w \in \mathbb{C} : |w| > (3/4)|z|\}$ to obtain, for a new constant $C < \infty$,

$$|v_x(z)|, |v_y(z)| \leqslant \frac{Ca}{|z|^3}.$$

So, for all $r \geqslant 2$ and $\theta \in (0, \pi)$,

$$|f(re^{i\theta})| = \left| \int_{re^{i\theta}}^\infty f'(z) dz \right| \leqslant \sqrt{2} Ca \int_r^\infty s^{-3} ds = \frac{Ca}{\sqrt{2} r^2}.$$

□

The constant a_K is called the *half-plane capacity* of K, written $\operatorname{hcap}(K)$. It follows from (1) that $\operatorname{hcap}(K) \geqslant 0$ and, for $K \subseteq K'$, $\operatorname{hcap}(K) \leqslant \operatorname{hcap}(K')$, with equality only if $K = K'$. Also, by uniqueness in Proposition 3.1, for any $r > 0$, $g_{rK}(z) = rg_K(z/r)$ and so $\operatorname{hcap}(rK) = r^2 \operatorname{hcap}(K)$. Similarly, for $b \in \mathbb{R}$, $g_{K+b}(z) = g_K(z-b) + b$ and $\operatorname{hcap}(K+b) = \operatorname{hcap}(K)$. Thus Proposition 3.2 adapts to all compact $\mathbb{H}$-hulls by scaling and translation. For the slit $S = (0, i]$, we have $g_S(z) = \sqrt{z^2+1}$ and $\operatorname{hcap}(S) = 1/2$. On the other hand, for the half-disc $A = \bar{\mathbb{D}} \cap \mathbb{H}$, $g_A(z) = z + z^{-1}$ and $\operatorname{hcap}(A) = 1$. By comparison with A, we see that $\operatorname{hcap}(K) \leqslant \operatorname{rad}(K)^2$ for all K, where $\operatorname{rad}(K)$ is the radius of the smallest ball centred on the real axis and containing K. By the estimate (2) and scaling, there is a constant $C < \infty$ such that, for all K and all $z \in \mathbb{H} \setminus K$,

$$|z - g_K(z)| \leqslant C \operatorname{rad}(K). \tag{3}$$

From the proof of Proposition 3.2, it is not hard to obtain

$$\operatorname{hcap}(K) = \lim_{y \to \infty} y \mathbb{E}_{iy}(\operatorname{Im}(B_T)).$$

Thus, half-plane capacity may be considered to measure the average height of the boundary of a hull seen by a Brownian motion started at ∞.

There is a useful identity for g_K in terms of the Brownian excursion E in$(\mathbb{H}, 0, \infty)$. Recall that this can be realised as the process $E_t = X_t + iR_t$, where $R_t = |W_t|$ and X and W are independent Brownian motions starting from 0 in $\mathbb{R}$ and $\mathbb{R}^3$ respectively.

Proposition 3.3. *Let K be a compact $\mathbb{H}$-hull and suppose that $H = \mathbb{H} \setminus K$ is a neighbourhood of 0. Then*

$$g_K'(0) = \mathbb{P}(E \text{ does not hit } K).$$

Proof. Let $Z = X + iY$ be a complex Brownian motion starting from $z = x + iy \in \mathbb{H}$. Define for $r \geqslant 0$

$$T_r = \inf\{t \geqslant 0 : Y_t = r\}, \quad T_K = \inf\{t \geqslant 0 : Z_t \in K\}.$$

Fix $r > y$ and set $M_t = y^{-1} Y_{T_0 \wedge T_r \wedge t}$. Then M is a bounded non-negative martingale with $M_0 = 1$ and with final value $Y_{T_0 \wedge T_r} = (r/y) 1_{\{T_0 > T_r\}}$. Define a new probability measure $\tilde{\mathbb{P}}$ by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = Y_{T_0 \wedge T_r}.$$

Note that, under $\tilde{\mathbb{P}}$, we have $T_0 > T_r$ almost surely. Moreover, under $\tilde{\mathbb{P}}$, by Girsanov's theorem and Lévy's characterization of Brownian motion,[3] X and Y remain independent, with X a Brownian motion and Y satisfying the stochastic differential equation

$$dY_t = dB_t + \frac{dt}{Y_t}, \quad t \leqslant T_r,$$

---

[3] See the course Stochastic Calculus and Applications

for some Brownian motion B. By Itô's formula, R also satisfies this equation, so, by uniqueness in law, if we start E from z, then the distributions of E and of Z under $\tilde{\mathbb{P}}$ agree up to T_r. Hence

$$\begin{aligned} p_r(z) &= \mathbb{P}_z(E \text{ does not hit } K \text{ before } T_r) \\ &= \mathbb{E}_z(y^{-1} Y_{T_0 \wedge T_r} 1_{\{T_K > T_r\}}) = (r/y)\mathbb{P}_z(T_r < T_0 \wedge T_K). \end{aligned}$$

Now $g_K(z) - z \to 0$ as $z \to \infty$ so, for r sufficiently large,

$$|\operatorname{Im} g_K(z) - r| \leqslant 1 \text{ whenever } \operatorname{Im}(z) = r,$$

and hence, by conformal invariance of Brownian motion,

$$\frac{\operatorname{Im} g_K(z)}{r+1} = \mathbb{P}_{g_K(z)}(T_{r+1} < T_0) \leqslant \mathbb{P}_z(T_r < T_0 \wedge T_K) \leqslant \mathbb{P}_{g_K(z)}(T_{r-1} < T_0) = \frac{\operatorname{Im} g_K(z)}{r-1}.$$

So

$$\mathbb{P}_z(E \text{ does not hit } K) = \lim_{r \to \infty} p_z(r) = \operatorname{Im} g_K(z)/y.$$

Now, $\operatorname{Im} g_K(z)/y \to g_K'(0) > 0$ as $z \to 0$ in $\mathbb{H}$. Take $\varepsilon > 0$ and set

$$S = \inf\{t \geqslant 0 : |E_t| = \varepsilon\},$$

then $|E_S| = \varepsilon$ and $\operatorname{Im} E_S > 0$ almost surely. Hence, choosing ε so that $K \cap \varepsilon\mathbb{D} = \emptyset$, by the strong Markov property of E and bounded convergence, as $\varepsilon \to 0$,

$$\mathbb{P}_0(E \text{ does not hit } K) = \mathbb{E}(\operatorname{Im} g_K(E_S)/\operatorname{Im}(E_S)) \to g_K'(0).$$

□

4 Loewner transforms

Consider a strictly increasing and unbounded family $(K_t)_{t \geqslant 0}$ of compact $\mathbb{H}$-hulls, starting from $\emptyset$. Set $K_{t,t+h} = g_{K_t}(K_{t+h} \setminus K_t)$. Say that $(K_t)_{t \geqslant 0}$ has the *local growth property* if

$$\operatorname{rad}(K_{t,t+h}) \to 0 \text{ as } h \downarrow 0, \text{ uniformly on compacts in } t. \tag{4}$$

For such a family, by compactness, there is, for each $t \geqslant 0$, a unique $\xi_t \in \mathbb{C}$ such that $\xi_t \in \overline{K_{t,t+h}}$ for all $h > 0$. Moreover, $\xi_t \in \mathbb{R}$ and, using the estimate (3), we can show that ξ_t depends continuously on t. The process $(\xi_t)_{t \geqslant 0}$ is called the *Loewner transform* of $(K_t)_{t \geqslant 0}$. Note that we have, for each $t \geqslant 0$, as $h \downarrow 0$,

$$\operatorname{hcap}(K_{t+h}) - \operatorname{hcap}(K_t) = \operatorname{hcap}(K_{t,t+h}) \leqslant \operatorname{rad}(K_{t,t+h})^2 \to 0.$$

So the map $t \mapsto \operatorname{hcap}(K_t)$ is a homeomorphism of $[0, \infty)$. By a time-reparametrization, we may if we wish assume that

$$\operatorname{hcap}(K_t) = 2t \quad \text{for all } t \geqslant 0. \tag{5}$$

Consider now any continuous real-valued function $(\xi_t)_{t\geqslant 0}$ and define a time-dependent vector field $b : [0,\infty) \times \mathbb{H} \to \mathbb{C}$ by

$$b(t,z) = \frac{2}{z-\xi_t} = \frac{2(x-\xi_t-iy)}{|z-\xi_t|^2}.$$

Then $b(t,.)$ is Lipschitz on $\{z \in \mathbb{H} : |z-\xi_t| \geqslant \varepsilon\}$, uniformly in $t \geqslant 0$, for all $\varepsilon > 0$. So, for each $z \in \mathbb{H}$, the differential equation

$$\dot{z}_t = b(t,z_t), \quad z_0 = z \tag{6}$$

has a unique maximal solution $(z_t)_{t<\zeta}$ in $\mathbb{H}$, with $\zeta \in (0,\infty]$. Moreover, if $\zeta < \infty$, then $\mathrm{Re}(z_t) - \xi_t \to 0$ and $\mathrm{Im}(z_t) \downarrow 0$ as $t \uparrow \zeta$. We write $z_t = g_t(z)$ and $\zeta = \zeta(z)$. Then $g_t(z)$ is defined whenever $t < \zeta(z)$. The family of maps $(g_t)_{t\geqslant 0}$ is the *solution flow* of the differential equation (6).

Lemma 4.1. *Set* $R_t = \max\{\sqrt{t}, \sup_{s\leqslant t} |\xi_s|\}$. *Then, for* $|z| \geqslant 4R_t$ *and* $s \leqslant t$, *we have* $|g_s(z) - z| \leqslant R_t$. *In particular* $\zeta(z) > t$.

Proof. Set $T = \inf\{s \geqslant 0 : |z_s - z| = R_t\} \wedge t$. For $s < T$, we have $|z_s - \xi_s| > 2R_t$ so $|\dot{z}_s| < 1/R_t$, and so $|z_T - z| < t/R_t \leqslant R_t$. Hence $T = t$. □

Proposition 4.2. *Let* $(K_t)_{t\geqslant 0}$ *be an increasing family of compact* $\mathbb{H}$*-hulls having the local growth property, parametrized so that* $\mathrm{hcap}(K_t) = 2t$ *for all* t. *Write* $(\xi_t)_{t\geqslant 0}$ *for the Loewner transform and set* $g_t = g_{K_t}$. *Then* $(g_t)_{t\geqslant 0}$ *is the solution flow of the differential equation* (6).

On the other hand, given any continuous real-valued function $(\xi_t)_{t\geqslant 0}$, *write* $(g_t)_{t\geqslant 0}$ *for the solution flow of* (6) *and set* $K_t = \{z \in \mathbb{H} : \zeta(z) \leqslant t\}$. *Then* $(K_t)_{t\geqslant 0}$ *is an increasing family of compact* $\mathbb{H}$*-hulls having the local growth property, with* $\mathrm{hcap}(K_t) = 2t$ *for all* t *and with Loewner transform* $(\xi_t)_{t\geqslant 0}$. *Moreover* $g_{K_t} = g_t$ *for all* t.

Proof. By Proposition 3.2, for any $\xi \in \bar{K} \cap \mathbb{R}$ and any $w \in \mathbb{H}$ with $|w-\xi| \geqslant 4\,\mathrm{rad}(K)$,

$$\left| g_K(w) - w - \frac{\mathrm{hcap}(K)}{w-\xi} \right| \leqslant \frac{2C\,\mathrm{rad}(K)\,\mathrm{hcap}(K)}{|w-\xi|^2}.$$

For $t, h \geqslant 0$, we apply this to $K = K_{t,t+h}$ with $w = g_t(z)$ to obtain

$$\left| g_{t+h}(z) - g_t(z) - \frac{2h}{g_t(z)-\xi_t} \right| \leqslant \frac{4C\,\mathrm{rad}(K_{t,t+h})h}{|g_t(z)-\xi_t|^2}$$

whenever $|g_t(z) - \xi_t| \geqslant 4\,\mathrm{rad}(K_{t,t+h})$. When combined with the local growth condition, this allows us to show that, for all $z \in H_t$, $g_t(z)$ is continuous and then differentiable in t with

$$\dot{g}_t(z) = 2/(g_t(z) - \xi_t).$$

On the other hand, starting from any continuous function $(\xi_t)_{t\geqslant 0}$, some standard applications of Gronwall's lemma[4] will show that $H_t = \{z \in \mathbb{H} : \zeta(z) > t\}$ is open, that $g_t : H_t \to \mathbb{H}$ is analytic, and that $z(g_t(z) - z) \to 2t$ as $|z| \to \infty$. To see that g_t is a bijection, note that, $\mathrm{Im}(b(s,z)) < 0$ for all s and z, so there is a unique solution $(z_s)_{0\leqslant s\leqslant t}$ to the differential equation (6) in $\mathbb{H}$ for any given terminal value $z_t \in \mathbb{H}$ and $g_t(z) = z_t$ if and only if $z = z_0$. Hence $g_t : H_t \to \mathbb{H}$ is a conformal isomorphism and, in particular, H_t is simply connected. By Lemma 4.1, $K_t = \mathbb{H} \setminus H_t$ is bounded and is therefore a compact $\mathbb{H}$-hull. Finally, a further application of Lemma 4.1, to $K_{t,t+h}$, shows that $(K_t)_{t\geqslant 0}$ satisfies

---

[4] Gronwall's lemma states that if f is an integrable function on the interval $[0,T]$ and if

$$f(t) \leqslant A + B\int_0^t f(s)ds$$

for all $t \in [0,T]$, for some constants $A, B \in [0,\infty)$, then $f(T) \leqslant Ae^{BT}$. This allows to show that the solution flow for a Lipschitz vector field v on $\mathbb{C}$ is continuous in the initial data as follows. Write $(g_t(z) : t \geqslant 0)$ for the unique solution to

$$\dot{g}_t(z) = v(g_t(z)), \quad g_0(z) = z.$$

Then, for $z, z' \in \mathbb{C}$ we have

$$|g_t(z) - g_t(z')| \leqslant |z - z'| + K\int_0^t |g_s(z) - g_s(z')|ds$$

for all $t \geqslant 0$, where K is the Lipschitz constant of v, so by Gronwall's lemma

$$|g_t(z) - g_t(z')| \leqslant |z - z'|e^{Kt}.$$

If v is moreover analytic, with second derivative bounded uniformly by K_2 say, then a further Gronwall argument allows to show that $(g_t(z) : t \geqslant 0)$ is analytic in z and that the derivative $z'_t = g'_t(z)$ satisfies

$$\dot{z}'_t = v'(g_t(z))z'_t, \quad z'_0 = 1,$$

so that $z'_t = \exp\int_0^t v'(g_s(z))ds$. To see this we note that, for $z, h \in \mathbb{C}$,

$$\begin{aligned}
|g_t(z+h) - g_t(z) - hz'_t| &= \left|\int_0^t (v(g_s(z+h)) - v(g_s(z)) - hv'(g_s(z))z'_s)ds\right| \\
&\leqslant \int_0^t |v(g_s(z+h)) - v(g_s(z)) - v'(g_s(z))(g_s(z+h) - g_s(z))|ds \\
&\quad + \int_0^t |v'(g_s(z))||g_s(z+h) - g_s(z) - hz'_s|ds \\
&\leqslant \frac{1}{2}K_2|h|^2 te^{2Kt} + K\int_0^t |g_s(z+h) - g_s(z) - hz'_s|ds
\end{aligned}$$

and apply Gronwall again. The above arguments extend without essential modification to a time-dependent vector field $v(t,z)$, analytic in z, provided K and K_2 can be chosen uniformly on compact time intervals. Moreover the arguments also localize, allowing to draw appropriately modified conclusions when v is analytic on a (possibly time-dependent) domain. We omit the details. In particular, if $\zeta(z) > t$ then $b(t,.)$ is Lipschitz on some space-time neighbourhood N of the path $(g_s(z) : s \leqslant t)$; the continuity estimate then shows that, for $\delta > 0$ sufficiently small, and $|z - z'| \leqslant \delta$, the path $(g_s(z') : s < t \wedge \zeta(z'))$ stays in N, so in fact $\zeta(z') > t$ also. This establishes that H_t is open.

the local growth condition. □

Let $(K_t)_{t\geqslant 0}$, $(g_t)_{t\geqslant 0}$ and $(\xi_t)_{t\geqslant 0}$ be as in Proposition 4.2, and write as usual $H_t = \mathbb{H}\setminus K_t$. Let I be an open interval in $\mathbb{R}$ containing ξ_0 and let N be a neighbourhood of I in $\mathbb{H}$. Note that $I \subseteq \hat{N}$. Suppose we are given a conformal isomorphism $\Phi : N \to N^*$, where $N^* \subseteq \mathbb{H}$ and such that $I^* = \Phi(I) \subseteq \mathbb{R}$. Then I^* is an open interval containing $\xi_0^* = \Phi(\xi_0)$ and N^* is a neighbourhood of I^* in $\mathbb{H}$. Set $\tilde{N} = \cup_{z\in N\cup I}\{z, \bar{z}\}$ and $\tilde{N}^* = \cup_{z\in N^*\cup I^*}\{z, \bar{z}\}$. We can extend Φ as a conformal isomorphism $\tilde{N} \to \tilde{N}^*$ by reflection, that is, by setting $\Phi(\bar{z}) = \overline{\Phi(z)}$ for $z \in N$.

Set $T = \inf\{t \geqslant 0 : \bar{K}_t \not\subseteq N \cup I\}$ and define, for $t < T$, $K_t^* = \Phi(K_t)$ and $H_t^* = \mathbb{H}\setminus K_t^*$. Note that K_t^* is a compact $\mathbb{H}$-hull, with $\bar{K}_t^* \subseteq N^* \cap I^*$. Set $a_t^* = \operatorname{hcap}(K_t^*)$ and $g_t^* = g_{K_t^*}$; set also $N_t = g_t(N \setminus K_t)$ and $N_t^* = g_t^*(N^* \setminus K_t^*)$, and finally $I_t = g_t(\hat{H}_t \setminus H_t \setminus (\mathbb{R}\setminus I))$ and $I_t^* = g_t^*(\hat{H}_t^* \setminus H_t^* \setminus (\mathbb{R}\setminus I^*))$. Then I_t and I_t^* are open intervals in $\mathbb{R}$, with $\xi_t \in I_t$. Moreover N_t and N_t^* are neighbourhoods in $\mathbb{H}$ of I_t and I_t^*, respectively.

Set $\Phi_t = g_t^* \circ \Phi \circ g_t^{-1}$. Then Φ_t is a conformal isomorphism $N_t \to N_t^*$, which extends analytically to I_t, with $\Phi(I_t) = I_t^*$. Moreover, we can extend Φ_t as a conformal isomorphism of reflected domains $\tilde{N}_t \to \tilde{N}_t^*$, in a manner analogous to that used for Φ above. Set $K_{t,t+h}^* = g_t^*(K_{t+h}^* \setminus K_t^*)$. Then $K_{t,t+h}^* = \Phi_t(K_{t,t+h})$ so, since $\overline{K_{t,t+h}} \subseteq N_t \cup I_t$ and using Lemma 2.5 to bound Φ_t', $(K_t^*)_{t<T}$ inherits the local growth property from $(K_t)_{t<T}$ and has Loewner transform given by $\xi_t^* = \Phi_t(\xi_t)$.

Proposition 4.3. *The map $t \mapsto a_t^*$ is differentiable on $[0, T)$ with $\dot{a}_t^* = 2\Phi_t'(\xi_t)^2$.*

Proof. We show first that $t \mapsto a_t^*$ is differentiable on the right at $t = 0$ with right derivative $2\Phi'(\xi_0)^2$. By scaling and translation properties of hcap, it suffices to consider the case where $\xi_0 = \xi_0^* = 0$ and where $\Phi'(0) = 1$. We obtain from (1) by scaling, for $r > 0$ and $K \subseteq r\mathbb{D}$,

$$\operatorname{hcap}(K) = \int_0^\pi \mathbb{E}_{re^{i\theta}}(\operatorname{Im}(B_{T(K)}))\left(\frac{2\sin\theta}{\pi}\right) r\,d\theta.$$

Multiply this formula by r and integrate over $r \in [r_1, r_2]$ to obtain

$$\frac{1}{2}(r_2^2 - r_1^2)\operatorname{hcap}(K) = \int_{A(r_1,r_2)} \mathbb{E}_z(\operatorname{Im}(B_{T(K)}))\left(\frac{2\operatorname{Im} z}{\pi}\right) A(dz),$$

where $A(dz)$ denotes area measure and

$$A(r_1, r_2) = \{z \in \mathbb{H} : r_1 \leqslant |z| \leqslant r_2\}.$$

Set $w = \Phi(z)$. By conformal invariance of Brownian motion,

$$\mathbb{E}_w(\operatorname{Im}(B_{T(K^*)})) = \mathbb{E}_z(\operatorname{Im}(B_{T(K)})).$$

Since $\Phi'(0) = 1$, given $\varepsilon > 0$, there exists $r > 0$ such that, for $r_1 \leqslant |w| \leqslant r_2 \leqslant r$,

$$A((1+\varepsilon)r_1, (1-\varepsilon)r_2) \subseteq \Phi^{-1}(A(r_1, r_2)) \subseteq A((1-\varepsilon)r_1, (1+\varepsilon)r_2),$$

$$1 - \varepsilon \leqslant \left|\frac{A(dw)}{A(dz)}\right| \leqslant 1 + \varepsilon,$$

$$(1-\varepsilon)\operatorname{Im}(z) \leqslant \operatorname{Im} w \leqslant (1+\varepsilon)\operatorname{Im}(z).$$

So, provided $K_t^* \subseteq r_1\mathbb{D}$, we have

$$\frac{1}{2}(r_2^2 - r_1^2)\,\mathrm{hcap}(K_t^*) = \int_{A(r_1,r_2)} \mathbb{E}_{\Phi^{-1}(w)}(\mathrm{Im}(\Phi(B_{T(K)}))) \left(\frac{2\,\mathrm{Im}\,w}{\pi}\right) A(dw),$$

and hence, by the above estimates,

$$(1-\varepsilon)^5\,\mathrm{hcap}(K_t) \leqslant \mathrm{hcap}(K_t^*) \leqslant (1+\varepsilon)^5\,\mathrm{hcap}(K_t).$$

Since $\mathrm{hcap}(K_t) = 2t$, and ε may be chosen arbitrarily small, this implies that $a_t^*/t \to 2$ as $t \downarrow 0$.

The same argument, applied to $K_{t,t+h}$, shows, for all $t < T$, that a^* is differentiable on the right at t with right derivative $2\Phi_t'(\xi_t)^2$. Since the right derivative is continuous in t, it follows that a^* is in fact differentiable with the claimed derivative. □

Proposition 4.4. *The map $t \mapsto \Phi_t$ on $[0,T)$ is a C^1 map of analytic functions, with*

$$\dot{\Phi}_t(\xi_t) = -3\Phi_t''(\xi_t), \quad \dot{\Phi}_t'(\xi_t) = \frac{1}{2}\frac{\Phi_t''(\xi_t)^2}{\Phi_t'(\xi_t)} - \frac{4}{3}\Phi_t'''(\xi_t).$$

Proof. By Propositions 4.2 and 4.3, $(g_t^*)_{t\leqslant t_0}$ satisfies

$$\dot{g}_t^*(z) = 2\Phi_t'(\xi_t)^2/(g_t^*(z) - \xi_t^*), \quad z \in H_t^*.$$

Set $f_t = g_t^{-1}$ and differentiate the equation $f_t(g_t(z)) = z$ in t to obtain

$$\dot{f}_t(z) = -2f_t'(z)/(z - \xi_t), \quad z \in \mathbb{H}.$$

Now $\Phi_t = g_t^* \circ \Phi \circ f_t$. We cannot take the derivative in t directly at the singularity ξ_t, but, by analyticity, it will suffice to differentiate, using the chain rule, at each $z \in N_t$, and then pass to the limit in the derivatives as $z \to \xi_t$. We have

$$\dot{\Phi}_t(z) = \dot{g}_t^*(\Phi(f_t(z)) + (g_t^*)'(\Phi(f_t(z)))\Phi'(f_t(z))\dot{f}_t(z) = \frac{2\Phi_t'(\xi_t)^2}{\Phi_t(z) - \Phi_t(\xi_t)} - \Phi_t'(z)\frac{2}{z - \xi_t}$$

and we can differentiate in z to obtain

$$\dot{\Phi}_t'(z) = 2\left(-\frac{\Phi_t'(\xi_t)^2\Phi_t'(z)}{(\Phi_t(z) - \Phi_t(\xi_t))^2} + \frac{\Phi_t'(z)}{(z - \xi_t)^2} - \frac{\Phi_t''(z)}{z - \xi_t}\right).$$

The desired identities may now be obtained by taking the limits in these equations as $z \to \xi_t$, using l'Hôpital's rule. □

5 SLE(κ)

Recall that, by Proposition 4.2, we can associate to any continuous function $\xi : [0, \infty) \to \mathbb{R}$, certain families of of compact $\mathbb{H}$-hulls K_t, domains $H_t = \mathbb{H} \backslash K_t$ and conformal isomorphisms $g_t : H_t \to \mathbb{H}$. We say that a continuous function $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ *generates* $(K_t)_{t \geqslant 0}$ if H_t is the connected component of $\mathbb{H} \setminus \gamma[0, t]$ containing ∞. Here $\gamma[0, t] = \{\gamma_s : s \in [0, t]\}$. Then we shall write $\operatorname{hcap}(\gamma[0, t])$ for $\operatorname{hcap}(K_t)$.

Fix $\kappa \in [0, \infty)$ and take $\xi = (\xi_t)_{t \geqslant 0}$ to be a (real) Brownian motion of diffusivity κ. Thus $\xi_t = \sqrt{\kappa} B_t$, where B is a (standard) Brownian motion. We refer to [1] for a proof of the following fundamental result.

Theorem 5.1. *For all $\kappa \in [0, \infty)$, there exists a unique continuous, non-self-traversing*[5] *random process $(\gamma_t)_{t \geqslant 0}$ in $\overline{\mathbb{H}}$ which generates $(K_t)_{t \geqslant 0}$.*

The process $(\gamma_t)_{t \geqslant 0}$ is called a *stochastic Loewner evolution of parameter* κ or *SLE*(κ) for short. The following result is an easy consequence of the Lévy–Khinchine theorem.

Proposition 5.2. *Let $\xi = (\xi_t)_{t \geqslant 0}$ be a continuous random process in $\mathbb{R}$. Then the following are equivalent:*

(a) *ξ is a Brownian motion of diffusivity κ for some $\kappa \in [0, \infty)$;*

(b) *ξ has stationary independent increments and the distribution of $(r^{-1} \xi_{r^2 t})_{t \geqslant 0}$ does not depend on $r \in (0, \infty)$.*

Given any continuous, non-self-traversing path γ, let us define, for $r > 0$, a *scaling operator* σ_r by

$$(\sigma_r \gamma)_t = r^{-1} \gamma_{r^2 t}.$$

Recall from Subsection 1.3 the definition of the shifted path $\theta_s \gamma$ from γ_s to ∞ in H_s.

It is straightforward to work out that the path $\sigma_r \gamma$ has Loewner transform $(r^{-1} \xi_{r^2 t})_{t \geqslant 0}$ and $g_s(\theta_s \gamma)$ has Loewner transform $(\xi_{s+t})_{t \geqslant 0}$. Hence we obtain the following result.

Proposition 5.3. *Let $\gamma = (\gamma_t)_{t \geqslant 0}$ be a continuous non-self-traversing random process in $\overline{\mathbb{H}}$, starting from 0, with $\operatorname{hcap}(\gamma[0, t]) = 2t$. Then the following are equivalent:*

(a) *γ is an SLE(κ) for some $\kappa \in [0, \infty)$;*

(b) *for all $r > 0$ and $s \geqslant 0$, both $\sigma_r \gamma$ and $g_s(\theta_s \gamma) - \xi_s$ have the same distribution as γ, and $g_s(\theta_s \gamma) - \xi_s$ is independent of $(\gamma_r)_{r \leqslant s}$.*

---

[5] We are using here a small extension of the notion of non-self-traversing path, as we do not yet know that $\gamma_t \to \infty$ as $t \to \infty$. This is however true, as we shall see in the next section.

6 Phases of SLE

Recall that one can scale a standard Brownian motion, either in time or space, to obtain a Brownian motion of any diffusivity. Thus "all Brownian motions look the same". The same is not true of SLE, as we shall shortly see. The Loewner flow extends continuously to $\bar{\mathbb{H}} \setminus \{0\}$. The following proposition then allows us to deduce information about the SLE curve γ from the lifetimes of the boundary Loewner flow[6].

Proposition 6.1. *Let $x \in \mathbb{R} \setminus \{0\}$ and let $t \geqslant 0$. Then $\zeta(x) > t$ if and only if $x \notin \bar{K}_t$.*

Proof. Suppose $\zeta(x) > t$, then by continuity of the flow $\zeta(z) > t$ for all $z \in N$, for some neighbourhood N of x in $\mathbb{H}$. But then $N \subseteq H_t$, so $x \notin \bar{K}_t$.

On the other hand suppose $x \notin \bar{K}_t$ and $x > 0$. By Proposition 3.1, for $s < \zeta(x) \wedge t$ we have

$$g_s(x) - \xi_s \geqslant g_{K_{s,t}}(g_s(x)) - g_{K_{s,t}}(\xi_s) = g_{K_t}(x) - g_{K_t}(\sup I_t) > 0$$

where $I_t = \bar{K}_t \setminus K_t$. Hence $\zeta(x) > t$. □

We shall need the following lemma concerning the solutions of the Bessel stochastic differential equation.

Lemma 6.2. *Fix $a \in (0, \infty)$ and consider, for each $x \in (0, \infty)$, the unique maximal solution $(X_t(x))_{t<\zeta(x)}$ in $(0, \infty)$ to*

$$dX_t = dB_t + \frac{a}{X_t}dt, \quad X_0 = x. \tag{7}$$

Then, for $x, y \in (0, \infty)$ with $x < y$, we have

- *for $a \in (0, 1/4]$, $\mathbb{P}(\zeta(x) < \zeta(y) < \infty) = 1$;*
- *for $a \in (1/4, 1/2)$, $\mathbb{P}(\zeta(x) < \infty) = 1$ and $\mathbb{P}(\zeta(x) < \zeta(y)) = \phi(\frac{y-x}{y})$;*
- *for $a \in [1/2, \infty)$, $\mathbb{P}(\zeta(x) < \infty) = 0$.*

Here, for $a \in (1/4, 1/2)$ and $\theta \in [0, 1]$, $\phi(\theta)$ is given by

$$\phi(\theta) \propto \int_0^\theta \frac{du}{u^{2-4a}(1-u)^{2a}}, \quad \phi(1) = 1.$$

Proof. First we shall show, for all $x > 0$, that

$$\mathbb{P}(\zeta(x) < \infty) = \begin{cases} 1, & \text{if } a < 1/2 \\ 0, & \text{if } a \geqslant 1/2. \end{cases}$$

---

[6] Actually we could avoid the need for this by improving Loewner's theorem to show that $\bar{K}_t = \{z \in \bar{\mathbb{H}} : \zeta(x) \leqslant t\}$.

Fix $x > 0$ and write $X_t = X_t(x)$ and $\zeta = \zeta(x)$. Assume for now that $a \neq 1/2$. Define, for $t < \zeta$, $M_t = X_t^{1-2a}$. By Itô's formula,

$$dM_t = (1-2a)X_t^{-2a}dX_t - a(1-2a)X_t^{-2a-1}dt = (1-2a)X_t^{-2a}dB_t,$$

so M is a local martingale. Consider, for $0 < r < R$, the stopping times

$$S = S(r) = \inf\{t \in [0,\zeta) : X_t \in \{r, R\}\}, \quad T = T(R) = S(0) = \inf\{t \in [0,\zeta) : X_t = R\}.$$

Assume that $r < x < R$, then $S(r) < \zeta$. Now M is bounded up to $S(r)$ so, by optional stopping[7],

$$x^{1-2a} = M_0 = \mathbb{E}(M_S) = r^{1-2a}\mathbb{P}(X_S = r) + R^{1-2a}\mathbb{P}(X_S = R). \tag{8}$$

For $a \in (0, 1/2)$, this argument applies also when $r = 0$, and we obtain

$$\mathbb{P}(T < \zeta) = (x/R)^{1-2a}.$$

Hence, on letting $R \to \infty$, we see that $\mathbb{P}(\zeta < \infty) = 1$. On the other hand, for $a \in (1/2, \infty)$, we obtain on letting $r \to 0$ that

$$\mathbb{P}(T < \zeta) = 1$$

and so, since $T(R) \to \infty$ as $R \to \infty$, we deduce that $\mathbb{P}(\zeta = \infty) = 1$. In the case $a = 1/2$, an argument similar to that used for $a > 1/2$ may be made using the process $M_t = \log X_t$.

Assume from now on that $a \in (0, 1/2)$. It remains to show that, for $0 < x < y$,

$$\mathbb{P}(\zeta < \zeta(y)) = \begin{cases} 1, & \text{if } a < 1/4, \\ \phi(\frac{y-x}{y}), & \text{if } a > 1/4. \end{cases}$$

Define for $\theta \in [0, 1]$

$$\chi(\theta) = \int_\theta^1 \frac{du}{u^{2-4a}(1-u)^{2a}}.$$

Note that χ is continuous on $[0, 1]$ as a map into $[0, \infty]$, with $\chi(0) < \infty$ for $a \in (1/4, 1/2)$ and $\chi(0) = \infty$ for $a \in (0, 1/4]$. Note also χ is C^2 on $(0, 1)$, with

$$\chi''(\theta) + 2\left(\frac{1-2a}{\theta} - \frac{a}{1-\theta}\right)\chi'(\theta) = 0.$$

Fix $y > x$ and write $Y_t = X_t(y)$. Define, for $t < \zeta$, $R_t = Y_t - X_t$, $\theta_t = R_t/Y_t$ and $N_t = \chi(\theta_t)$. By Itô's formula,

$$dR_t = -\frac{aR_t dt}{X_t Y_t}, \quad d\theta_t = \left(\frac{\theta_t}{Y_t}\right)^2 \left(\frac{1-2a}{\theta_t} - \frac{a}{1-\theta_t}\right)dt - \frac{\theta_t}{Y_t}dB_t,$$

---

[7] By the almost-sure martingale convergence theorem, M_t converges as $t \uparrow S$. On the event $S = \infty$, this implies that X_t converges as $t \to \infty$, and hence $[M]_t = (1-2a)^2 \int_0^t X_s^{-4a} ds \to \infty$, a contradiction. Hence $\mathbb{P}(S < \infty) = 1$.

so

$$dN_t = \chi'(\theta_t)d\theta_t + \frac{1}{2}\chi''(\theta_t)d\theta_t d\theta_t = -\frac{\chi'(\theta_t)\theta_t dB_t}{Y_t}.$$

Hence N is a local martingale.

Consider the random variables

$$A(x) = \int_0^\zeta \frac{1}{X_t^2}dt, \quad A_n = \int_{T(2^{-n+1}x)}^{T(2^{-n}x)} \frac{1}{X_t^2}dt, \quad n \geqslant 1.$$

By the strong Markov property and a scaling argument, the positive random variables A_n are independent and identically distributed, so $A(x) = \infty$ almost surely.

Since N is non-negative, both N_t and the quadratic variation

$$[N]_t = \int_0^t \frac{\chi'(\theta_s)^2\theta_s^2}{Y_s^2}ds$$

converge to a finite limit almost surely as $t \uparrow \zeta$. Hence, in particular, θ_t converges as $t \uparrow \zeta$. If $\zeta < \zeta(y)$, then $\theta_\zeta = 1$ so $N_\zeta = 0$. If $\zeta = \zeta(y)$, then the conjunction of $A(y) = \infty$ and $[N]_\zeta < \infty$ forces $\theta_t \to 0$ as $t \uparrow \zeta$. In the case $a \in (0, 1/4]$, this would imply that $N_t = \chi(\theta_t) \to \infty$ as $t \uparrow \zeta$, a contradiction, so $\mathbb{P}(\zeta < \zeta(y)) = 1$. On the other hand, for $a \in (1/4, 1/2)$, N is bounded up to ζ so, by optional stopping,

$$\chi\left(\frac{y-x}{y}\right) = N_0 = \mathbb{E}(N_\zeta) = \chi(0)\mathbb{P}(\zeta = \zeta(y)).$$

□

Proposition 6.3. *Let γ be an SLE(κ), with $\kappa \in [0, 4]$. Then $\gamma(0, \infty) \subseteq \mathbb{H}$ and γ is a simple curve, almost surely.*

Proof. For $\kappa = 0$, we have $\gamma_t = \sqrt{2t}i$ for all t, so the claim holds. Fix $x \in (0, \infty)$ and set $W_t = -\xi_t/\sqrt{\kappa}$, $a = 2/\kappa$ and $X_t = W_t + g_t(x)/\sqrt{\kappa} = (g_t(x) - \xi_t)/\sqrt{\kappa}$. Then W is a standard Brownian motion, X satisfies the Bessel stochastic differential equation

$$dX_t = dW_t + \frac{a}{X_t}dt, \quad X_0 = \frac{x}{\sqrt{\kappa}},$$

and $\zeta(x) = \inf\{t \geqslant 0 : X_t = 0\}$. Hence, by Lemma 6.2, $\zeta(x) = \infty$ almost surely[8]. Since $\zeta(x) \leqslant \zeta(y)$ whenever $0 < x \leqslant y$, we deduce that almost surely, for all $x > 0$, and then for all $x < 0$ by symmetry, $\zeta(x) = \infty$. We have shown that $\gamma(0, \infty) \subseteq \mathbb{H}$ almost surely. If $\gamma_t = \gamma_{t'}$ for some $t < t'$, then for any $s \in (t, t')$, we have $(\theta_s\gamma)_{t'-s} = g_s(\gamma_{t'}) - \xi_s \in \mathbb{R}$. In particular, $(\theta_s\gamma)(0, \infty) \cap \mathbb{R}$ is non-empty for some rational $s \geqslant 0$, an event of probability zero. □

Proposition 6.4. *Let γ be an SLE(κ), with $\kappa \in [0, 4]$. Then $\gamma_t \to \infty$ as $t \to \infty$, almost surely.*

---

[8] The relevant quantity in the notation of Lemma 6.2 is $\zeta(X_0) = \zeta(x/\sqrt{\kappa})$.

Proof for $\kappa \in [0,4)$. Fix $r \in (0,1)$ and set $\tau = \tau_r = \inf\{t \geqslant 0 : |\gamma_t - 1| = r\}$. We shall show shortly, using a harmonic measure estimate, that $g_\tau(1) - \xi_\tau \leqslant 2r$. On the other hand, by letting $R \to \infty$ and then $r \to 0$ in equation (8), we see that in fact $\inf_{t\geqslant 0}(g_t(1) - \xi_t) > 0$ almost surely. So we must have, almost surely, $\tau_r = \infty$ for some r.

Condition on γ and suppose that $\tau < \infty$. Let $(W_t)_{t\geqslant 0}$ be a complex Brownian motion starting from $z = iy$. Set $W'_t = g_\tau(W_t)$ and $z' = g_\tau(z)$. Write I for the complex line segment of length r from γ_τ to 1, and write I' for the real interval $[g_\tau(\gamma_\tau), g_\tau(1)]$. Set

$$T = \inf\{t \geqslant 0 : W_t \in \gamma[0,\tau] \cup I \cup \mathbb{R}\}, \quad T' = \inf\{t \geqslant 0 : W'_t \in \mathbb{R}\}.$$

Note that $z'/z \to 1$ as $y \to \infty$ and, since K_τ is connected, $W'_{T'} \in I'$ only if $W_T \in I$. Then, by conformal invariance of Brownian motion,

$$g_\tau(1) - \xi_\tau = |I'| = \lim_{y\to\infty} \frac{y}{\pi}\mathbb{P}(W'_{T'} \in I') \leqslant \lim_{y\to\infty} \frac{y}{\pi}\mathbb{P}(W_T \in I) \leqslant 2r.$$

Now, a similar argument shows that, almost surely,

$$\operatorname{dist}(g_1(\gamma[1,\infty)), \{g_1(0-), g_1(0+)\}) > 0$$

so $\operatorname{dist}(\gamma[1,\infty), 0) > 0$, and so $\liminf_{t\to\infty} |\gamma(t)|$ is positive, and hence is infinite by scaling. □

Proposition 6.5. *Let γ be an SLE(κ), with $\kappa = 2/a \in (4,8)$. Then, for any $x, y \in (0,\infty)$ with $x < y$,*

$$\mathbb{P}(\gamma \text{ hits } [x,y]) = \phi\left(\frac{y-x}{y}\right).$$

Moreover, almost surely, γ is not a simple curve, nor a space-filling curve, but $\operatorname{dist}(0, H_t) \to \infty$ *as $t \to \infty$.*

Proof. Set

$$X_t = \frac{g_t(x) - \xi_t}{\sqrt{\kappa}}, \quad Y_t = \frac{g_t(y) - \xi_t}{\sqrt{\kappa}}, \quad W = -B.$$

Then

$$dX_t = dW_t + \frac{a}{X_t}dt, \quad dY_t = dW_t + \frac{a}{Y_t}dt, \quad X_0 = \frac{x}{\sqrt{\kappa}}, \quad Y_0 = \frac{y}{\sqrt{\kappa}}.$$

So, by Lemma 6.2,

$$\mathbb{P}(\gamma \text{ hits } [x,\infty)) = \mathbb{P}(X \text{ hits } 0) = 1,$$

and

$$\mathbb{P}(\gamma \text{ hits } [x,y]) = \mathbb{P}(\zeta(x) < \zeta(y)) = \phi\left(\frac{y-x}{y}\right) \in (0,1).$$

Hence $\gamma_{\zeta(x)} \in (x,\infty)$ almost surely. Moreover, for $y > x$, we have $\{\gamma_{\zeta(x)} < y\} = \{\zeta(y) > \zeta(x)\}$ and $\{\gamma_{\zeta(x)} \geqslant y\} = \{\zeta(y) = \zeta(x)\}$ and both events have positive probability. In particular, we see that γ hits any given interval in $\mathbb{R}$ of positive length with positive probability. Now $g_1(\partial K_1 \cap \mathbb{H})$ is an interval of positive length, so is hit by $(g_1(\gamma_t))_{t>1}$ with positive probability; but $\partial K_1 \cap \mathbb{H} \subseteq \gamma[0,t]$, so γ has double points with positive probability and hence almost surely by a zero-one argument. On the other hand, on $\{\gamma_{\zeta(x)} > y\}$, there is a neighbourhood of $[x,y]$ in $\mathbb{H}$ which does not meet γ and $\operatorname{dist}([x,y], H_{\zeta(x)}) > 0$. In particular, γ is not space-filling, with positive probability, and then almost surely.

The set S of limit points of $g_{\zeta(1)}(z)$ as $z \to 0$, $z \in \mathbb{H}$ is a compact (possibly empty) subset of $(-\infty, \xi_{\zeta(1)})$. Pick $y < \inf S$. With positive probability, $\operatorname{dist}(S, g_{\zeta(1)}(H_{\zeta(y)})) > 0$, so $\operatorname{dist}(0, H_{\zeta(y)}) > 0$, so $\mathbb{P}(\operatorname{dist}(0, H_t) > 0) = \delta$ for some $t > 0$ and $\delta > 0$. This extends to all t by scaling, with the same δ. So $\mathbb{P}(\operatorname{dist}(0, H_t) > 0 \text{ for all } t > 0) = \delta$ and then $\delta = 1$ by a zero-one argument. Finally $\operatorname{dist}(0, H_t)$ is non-decreasing and, for all $r < \infty$, as $t \to \infty$,

$$\mathbb{P}(\operatorname{dist}(0, H_t) \leqslant r) = \mathbb{P}(\operatorname{dist}(0, H_1) \leqslant r/\sqrt{t}) \to 0.$$

□

Here is an elaboration of the zero-one argument for double points. Define, for $t > 0$, $A_t = \{\gamma_s = \gamma_{s'} \text{ for some distinct } s, s' \in [0,t]\}$. Then the sets A_t are non-decreasing in t and all have the same probability, p say, by scaling. But then $p = \mathbb{P}(\cap_t A_t)$ and $\cap_t A_t \in \mathcal{F}_{0+}$, where $\mathcal{F}_{0+} = \cap_{t>0} \sigma(\xi_s : s \leqslant t)$. But, by Blumenthal's zero-one law, $\mathcal{F}_{0+}$ contains only null sets and their complements. Hence $p \in \{0,1\}$.

We can deduce from Lemma 6.2 that, for $\kappa \in [8,\infty)$, we have $\gamma_{\zeta(x)} = x$ for all $x \in \mathbb{R}$, so $\mathbb{R} \subseteq \gamma[0,\infty)$. A proof of the following stronger result is given in [1].

Proposition 6.6. *Let γ be an SLE(κ), with $\kappa \in [8,\infty)$. Then, almost surely, $\gamma[0,\infty) = \mathbb{H}$. In particular, γ is space-filling and $\gamma_t \to \infty$ as $t \to \infty$.*

7 SLE and the domain Markov property

Recall from Section 1 that, for a domain $D = (U, z_0, z_1) \in \mathcal{D}$, we write P_D for the set of paths in D and C_D for the set of chords in D. Let $\mathcal{P}_D$ and $(\mathcal{P}_{D,t})_{t\in[0,1]}$ denote the usual σ-algebra and filtration, respectively, on P_D, generated by the coordinate maps $\gamma \mapsto \gamma_t$. We can then define a σ-algebra on C_D by

$$\mathcal{C}_D = \{A \subseteq C_D : \{\gamma : [\gamma] \in A\} \in \mathcal{P}_D\}.$$

For paths γ, γ' in D with $\gamma \sim \gamma'$, for $n \geqslant 1$ and for Borel sets $B_1, \dots, B_n \subseteq \hat{U}$

$$\begin{aligned} &\gamma_{t_1} \in B_1, \dots, \gamma_{t_n} \in B_n \text{ for some } t_1 < \dots < t_n \\ &\qquad \Leftrightarrow \gamma'_{s_1} \in B_1, \dots, \gamma'_{s_n} \in B_n \text{ for some } s_1 < \dots < s_n. \end{aligned}$$

So we may consistently define

$$A(B_1, \dots, B_n) = \{[\gamma] \in C_D : \gamma_{t_1} \in B_1, \dots, \gamma_{t_n} \in B_n \text{ for some } t_1 < \dots < t_n\}.$$

Then $A(B_1, \dots, B_n) \in \mathcal{C}_D$ and $\mathcal{C}_D$ is generated by sets of this form.

We refer to Subsection 1.3 for the definition of the domain Markov property. Now that we have the filtration $(\mathcal{P}_{D,t})_{t\in[0,1]}$ and the σ-algebra $\mathcal{C}_D$, the notions of a stopping time on P_D and of a measure on C_D are fully specified. As usual, τ is a stopping time if $\{\gamma : \tau(\gamma) \leqslant t\} \in \mathcal{P}_{D,t}$ for all $t \in [0, 1]$.

Theorem 7.1. *For $D \in \mathcal{D}$, let μ_D be a measure on $(C_D, \mathcal{C}_D)$. The following statements are equivalent:*

(a) *the family $(\mu_D : D \in \mathcal{D})$ is conformally invariant and has the domain Markov property; also the following regularity condition holds: if $X \sim \mu_{\mathbb{H},0,\infty)}$ and γ is a representative of X, parametrized by $[0,\infty]$, then $t \mapsto \operatorname{hcap}(\gamma[0,t])$ is a homeomorphism of $[0,\infty]$;*

(b) *there exists $\kappa \in [0,\infty)$ such that, for all $D \in \mathcal{D}$ and any conformal isomorphism $\Phi_D : (\mathbb{H}, 0, \infty) \to D$, μ_D is the law of $[\Phi_D(\gamma)]$, where γ is an SLE(κ).*

Proof. Suppose that (a) holds and that $X \sim \mu_{(\mathbb{H},0,\infty)}$. By the regularity condition we can choose a representative $(\gamma_t)_{t\geqslant 0}$ of X such that $\operatorname{hcap}(\gamma[0,t]) = 2t$ for all t. By scale invariance and the domain Markov property, $(\gamma_t)_{t\geqslant 0}$ satisfies condition (b) of Proposition 5.3 and hence is an SLE(κ) for some κ. Hence we obtain (b) by conformal invariance.

Suppose on the other hand that (b) holds. By scaling of SLE, the law of $[\Phi_D(\gamma)]$ does not depend on the choice of Φ_D and the family of measures $(\mu_D : D \in \mathcal{D})$ is conformally invariant. Finally, the strong Markov property of Brownian motion implies that $\theta_\tau\gamma \sim \gamma$ for any stopping time τ, so we obtain the domain Markov property. ☐

When (b) holds, we shall refer to μ_D and to any random chord $X \sim \mu_D$ as SLE(κ) in D. The regularity condition used in this result can certainly be replaced by something weaker at the cost of a more involved argument.

8 SLE(6) and the locality property

Consider a family $(\mu_D : D \in \mathcal{D})$, where μ_D is a probability measure on $(C_D, \mathcal{C}_D)$ for each $D \in \mathcal{D}$. Recall that $(\mu_D : D \in \mathcal{D})$ has the *locality property* if, for all $D, D' \in \mathcal{D}$, for all initial domains N common to D and D', for $X \sim \mu_D$ and $X' \sim \mu_{D'}$, we have $X^N \sim X'^N$, where X^N is X stopped on hitting the cut. When $(\mu_D : D \in \mathcal{D})$ is conformally invariant, this property is equivalent to the following property of $\mu = \mu_{(\mathbb{H},0,\infty)}$: let N and N^* be initial domains in $(\mathbb{H}, 0, \infty)$ and suppose we are given a conformal isomorphism $\Phi : N \to N^*$, taking 0 to 0, and $\partial N \cap \mathbb{R}$ onto $\partial N^* \cap \mathbb{R}$; if $X \sim \mu$, then $X^{N^*} \sim \Phi(X^N)$.

Theorem 8.1. *SLE*(6) *has the locality property.*

Proof. Let γ be an SLE(6) and let $\Phi : N \to N^*$ be an isomorphism of initial domains in $(\mathbb{H}, 0, \infty)$, as above. Set $T = \inf\{t \geqslant 0 : \gamma_t \notin N \cup (\partial N \cap \mathbb{R})\}$. For $t \leqslant T$, let $K_t^* = \Phi(K_t)$, $a_t^* = \operatorname{hcap}(K_t^*)$, $g_t^* = g_{K_t^*}$ and $\Phi_t = g_t^* \circ \Phi \circ g_t^{-1}$. Then $(K_t^*)_{t\leqslant T}$ has the local growth property, with Loewner transform given by $\xi_t^* = \Phi_t(\xi_t)$. Moreover, by Proposition 4.3, the map $t \mapsto a_t^*$ is differentiable, with $\dot{a}_t^* = 2\Phi_t'(\xi_t)^2$. Define τ_s and $\tilde{T}$ by

$$s = \int_0^{\tau_s} \Phi_t'(\xi_t)^2 dt, \qquad \text{for } s \leqslant T = \tau_{\tilde{T}}$$

and set $\tilde{K}_s = K_{\tau_s}^*$, $\tilde{g}_s = g_{\tau_s}^*$ and $\tilde{\xi}_s = \xi_{\tau_s}^*$. Then $\operatorname{hcap}(\tilde{K}_s) = 2s$ and $(\tilde{\xi}_s)_{s \leqslant \tilde{T}}$ is the Loewner transform of $(\tilde{K}_s)_{s \leqslant \tilde{T}}$. So, by Proposition 4.2, we have, for all $z \in \tilde{H}_s$,

$$\dot{\tilde{g}}_s(z) = \frac{2}{\tilde{g}_s(z) - \tilde{\xi}_s}.$$

Now, by Itô's formula and Proposition 4.4,

$$d\xi_t^* = \dot{\Phi}_t(\xi_t)dt + \Phi_t'(\xi_t)d\xi_t + \frac{1}{2}\Phi_t''(\xi_t)\kappa dt = \left(\frac{\kappa}{2} - 3\right)\Phi_t''(\xi_t)dt + \Phi_t'(\xi_t)d\xi_t.$$

Hence, for $\kappa = 6$, $(\tilde{\xi}_s)_{s \leqslant \tilde{T}}$ is a Brownian motion of diffusivity 6. Thus $(\Phi(\gamma_{\tau_s}))_{s \leqslant \tilde{T}}$ is an SLE(6), as required. ☐

Corollary 8.2. *Let U be a simply connected proper domain and let z_0, z_1, z_1' be distinct points in $\hat{U}$. Set $D = (U, z_0, z_1)$ and $D' = (U, z_0, z_1')$. Let X be SLE(6) in D and let X' be an SLE(6) in D'. Then X^T and $(X')^{T'}$ have the same distribution, where T and T' are, respectively, the first times that X and X' hit the boundary segment in $\hat{U}$ from z_1 to z_1'.*

9 SLE(8/3) and the restriction property

Recall that, for $D \in \mathcal{D}$, by a *filling* in D, we mean a closed, connected, simply connected subset of $\hat{U}$ containing z_0 and z_1. Write S_D for the set of such fillings. The family of sets $S_{\tilde{D}}$ with $\tilde{D} \subseteq D$ is a π-system on S_D. We denote the σ-algebra generated by this π-system by $\mathcal{S}_D$. Consider a family $(\mu_D : D \in \mathcal{D})$, where μ_D is a probability measure on $(S_D, \mathcal{S}_D)$ for each $D \in \mathcal{D}$. Recall from Subsection 1.1 that $(\mu_D : D \in \mathcal{D})$ has the *restriction property* if, for all $D, D' \in \mathcal{D}$ with $D' \subseteq D$, the conditional law of a random filling $X \sim \mu_D$, given $X \subseteq D'$ is $\mu_{D'}$. When $(\mu_D : D \in \mathcal{D})$ is conformally invariant, this property is equivalent to the following property of $\mu = \mu_{(\mathbb{H},0,\infty)}$: for all $D \subseteq (\mathbb{H}, 0, \infty)$, if $X \sim \mu$ and if $\Phi : D \to (\mathbb{H}, 0, \infty)$ is a conformal isomorphism, then, conditional on $X \subseteq D$, also $\Phi(X) \sim \mu$.

Proposition 9.1. *Let γ be an SLE(8/3). Then, for all $D \subseteq (\mathbb{H}, 0, \infty)$, we have $\mathbb{P}(\gamma \subseteq D) = \Phi_D'(0)^{5/8}$, where Φ_D is the unique conformal isomorphism $D \to \mathbb{H}$ with $\Phi_D(0) = 0$ and $\Phi_D(z) \sim z \to 0$ as $z \to \infty$.*

Proof. Note that it will suffice to prove the result when $\mathbb{H} \backslash D$ is covered by balls of uniformly positive radius, with at least half of each ball contained in $\mathbb{H}$. For general D, this implies that $\mathbb{P}(\gamma \subseteq D_\varepsilon) = \Phi'_{D_\varepsilon}(0)^{5/8}$ for all $\varepsilon > 0$, where D_ε is the set of points in D which are at distance at least ε from $\mathbb{H} \setminus D$. On letting $\varepsilon \to 0$ we obtain the general formula.

Write $\kappa = 8/3$ and $\alpha = 5/8$. Define $\Phi_t = g^*_t \circ \Phi_D \circ g_t^{-1}$, as above and set $\Sigma_t = \Phi'_t(\xi_t)$ and $M_t = \Sigma_t^\alpha$. By Itô's formula and Proposition 4.4,

$$d\Sigma_t = \dot{\Phi}'_t(\xi_t)dt + \Phi''_t(\xi_t)d\xi_t + \frac{1}{2}\Phi'''_t(\xi_t)\kappa dt = \Phi''_t(\xi_t)d\xi_t + \frac{1}{2}\frac{\Phi''_t(\xi_t)^2}{\Phi'_t(\xi_t)}dt + \left(\frac{\kappa}{2} - \frac{4}{3}\right)\Phi'''_t(\xi_t)dt.$$

Note that, since $\kappa = 8/3$, the final term vanishes. Also, by Itô's formula,

$$dM_t = \alpha\Sigma_t^{\alpha-1}d\Sigma_t + \frac{1}{2}\alpha(\alpha-1)\Sigma_t^{\alpha-2}d\Sigma_t d\Sigma_t = \alpha M_t dY_t,$$

where

$$dY_t = \frac{d\Sigma_t}{\Sigma_t} + \frac{1}{2}(\alpha-1)\frac{\Phi''_t(\xi_t)^2}{\Sigma_t^2}\kappa dt = \frac{\Phi''_t(\xi_t)}{\Sigma_t}d\xi_t + \frac{1}{2}(1+(\alpha-1)\kappa)\frac{\Phi''_t(\xi_t)^2}{\Sigma_t^2}dt.$$

Since $\kappa = 8/3$ and $\alpha = 5/8$, the final term vanishes, so Y and hence also M is a local martingale.

Now $\Phi'_t(x) \in [0,1]$ for all $x \in \partial(g_t(D)) \cap \mathbb{R}$. Hence M is a martingale up to the stopping time $T = \inf\{t \geqslant 0 : \gamma_t \notin D\}$. By Proposition 3.3, conditional on γ, we have

$$\Phi'_t(\xi_t) = \mathbb{P}_{\xi_t}(E \text{ does not leave } g_t(D)).$$

On the event $\{T = \infty\}$, since $\limsup_{t\to\infty} \operatorname{Im}(\gamma_t) = \infty$ almost surely, we must have $\Phi'_t(\xi_t) \to 1$ as $t \uparrow \infty$. On the other hand, on $\{T < \infty\}$, γ_T lies on the boundary of a ball B of positive radius, at least half of which is contained in $\mathbb{H} \setminus D$. There is a constant $\alpha > 0$ such that a Brownian motion started from any point in $B \cap \mathbb{H}$ hits both boundary segments $[-\infty, \gamma_T]$ and $[\gamma_T, \infty]$ of H_T with probability exceeding α. By conformal invariance of Brownian motion, this implies that $g_T(B \cap \mathbb{H})$ is contained in a proper cone with apex at ξ_T. Hence, conditional on γ and on $\{T < \infty\}$, starting from ξ_T, E hits $g_T(B \cap \mathbb{H})$ and hence leaves $g_T(D)$ almost surely, so $\Phi'_T(\xi_T) = 0$. Hence, by optional stopping,

$$\Phi'_D(0)^{5/8} = M_0 = \mathbb{E}(M_T) = \mathbb{P}(\gamma \subseteq D).$$

□

Theorem 9.2. *SLE(8/3) has the restriction property.*

Proof. Let γ be an SLE(8/3) and let $D, D' \subseteq D_0 = (\mathbb{H}, 0, \infty)$. Set $\tilde{D} = \Phi_D^{-1}(D')$. Write μ for the law of γ on $\mathcal{S}_{D_0}$ and $\mu(.|D)$ for the conditional law on $\mathcal{S}_{D_0}$ of $\Phi_D(\gamma)$ given $\gamma \subseteq D$. Then

$$\mathbb{P}(\Phi_D(\gamma) \subseteq D') = \mathbb{P}(\gamma \subseteq \tilde{D}) = \Phi'_{\tilde{D}}(0)^{5/8} = \Phi'_{D'}(0)^{5/8}\Phi'_D(0)^{5/8} = \mathbb{P}(\gamma \subseteq D')\mathbb{P}(\gamma \subseteq D),$$

so $\mu(S_{D'}|D) = \mu(S_{D'})$. Since D' was arbitrary, it follows that $\mu = \mu(.|D)$ by uniqueness of extension. □

10 Cardy's formula and Smirnov's theorem

Cardy derived a remarkable formula for the limits of crossing probabilities of critical percolation models. This gave much impetus to the search for scaling limits for percolation and other models, which are now thought, and in some cases known, to be given by SLE. For SLE itself, Cardy's formula becomes an exact calculation, of which the both of the following theorems are a generalization. The first is simply a restatement of Lemma 6.2.

Theorem 10.1. *Let γ be an SLE(κ), with $\kappa = 2/a \in (4,8)$, and let $x, y \in (0,\infty)$. Then*

$$\mathbb{P}(\gamma \textit{ hits } [x,y]) = \phi\left(\frac{y-x}{y}\right),$$

where ϕ is given by

$$\phi(\theta) \propto \int_0^\theta \frac{du}{u^{2-4a}(1-u)^{2a}}, \quad \theta \in [0,1], \quad \phi(1) = 1.$$

Theorem 10.2. *Let γ be an SLE(κ), with $\kappa = 2/a \in (4,\infty)$, and let $x, y \in (0,\infty)$. Then*

$$\mathbb{P}(\zeta(x) < \zeta(-y)) = \psi\left(\frac{y}{x+y}\right),$$

where ψ is given by

$$\psi(\theta) \propto \int_0^\theta \frac{du}{u^{2a}(1-u)^{2a}}, \quad \theta \in [0,1], \quad \psi(1) = 1.$$

Proof. Note that ψ is continuous on $[0,1]$ and is C^2 on $(0,1)$ with

$$\psi''(\theta) + 2a\left(\frac{1}{\theta} - \frac{1}{1-\theta}\right)\psi'(\theta) = 0.$$

Define, for $t \leqslant T = \zeta(x) \wedge \zeta(-y)$,

$$X_t = \frac{g_t(x) - \xi_t}{\sqrt{\kappa}}, \quad Y_t = -\frac{g_t(-y) - \xi_t}{\sqrt{\kappa}}, \quad R_t = X_t + Y_t, \quad \theta_t = \frac{Y_t}{R_t}$$

and set $M_t = \psi(\theta_t)$. Then, by Itô's formula,

$$dX_t = dB_t + \frac{a}{X_t}dt, \quad dY_t = -dB_t + \frac{a}{Y_t}dt, \quad dR_t = \frac{aR_t}{X_tY_t}dt,$$

so

$$d\theta_t = \frac{a}{R_t^2}\left(\frac{1}{\theta_t} - \frac{1}{1-\theta_t}\right)dt - \frac{dB_t}{R_t}$$

and so

$$dM_t = \psi'(\theta_t)d\theta_t + \frac{1}{2}\psi''(\theta_t)d\theta_t d\theta_t = -\frac{\psi'(\theta_t)dB_t}{R_t}.$$

We note that M is a local martingale, bounded up to T, and that $M_T = \theta_T$, with $\theta_T = 1$ if $\zeta(x) < \zeta(-y)$ and $\theta_T = 0$ if $\zeta(-y) < \zeta(x)$. Hence, by optional stopping,

$$\mathbb{P}(\zeta(x) < \zeta(-y)) = \mathbb{P}(\theta_T = 1) = \mathbb{E}(M_T) = M_0 = \psi(\theta_0) = \psi\left(\frac{y}{x+y}\right).$$

$\square$

Theorem 10.3 (Carleson). *Let γ be $SLE(6)$ in $(\Delta, 0, e^{\pi i/3})$, where Δ denotes the triangle with vertices $0, 1, e^{\pi i/3}$. Then the point X at which γ hits the edge $[1, e^{\pi i/3}]$ is uniformly distributed.*

Proof. The Schwarz–Christoffel transformation $(\mathbb{H}, 0, 1, \infty) \to (\Delta, 0, 1, e^{\pi i/3})$ is given by

$$f(z) = c\int_0^z \frac{dw}{w^{2/3}(1-w)^{2/3}}, \quad c = \frac{\Gamma(2/3)}{\Gamma(1/3)^2}.$$

In particular, for $y \geqslant 0$, $f(1+y) = 1 + e^{2\pi i/3}x$, where $x = \phi(\frac{y}{1+y})$. So, by conformal invariance and Cardy's formula,

$$\mathbb{P}(X \leqslant x) = \mathbb{P}(SLE(6) \text{ in } (\mathbb{H}, 0, \infty) \text{ hits } [1, 1+y]) = \phi\left(\frac{y}{1+y}\right) = x.$$

$\square$

We now discuss Smirnov's proof of Cardy's formula for percolation on the triangular lattice. Consider the lattice of edge length δ. Sites of the lattice are coloured black or white independently with probability 1/2. Take any Jordan domain D with three distinct boundary points $a(1), a(\tau), a(\tau^2)$, ordered positively, where $\tau = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Write Φ for the unique conformal isomorphism from D to the triangle Δ with corresponding boundary points $1, \tau, \tau^2$. For $z \in D$ and $\alpha \in \{1, \tau, \tau^2\}$, write $Q_\alpha(z)$ for the event that z is separated from the boundary segment $a(\tau\alpha)a(\tau^2\alpha)$ by a simple black path from $a(\alpha)a(\tau\alpha)$ to $a(\tau^2\alpha)a(\alpha)$. Set $H_\alpha(z) = H_\alpha^\delta(z) = \mathbb{P}(Q_\alpha(z))$. By a black path we mean any path in the lattice which visits only black points. The functions $H_\alpha(z)$ are constant in the interior of lattice triangles with discontinuities at the edges. Let f_α denote the unique affine function on Δ with $f_\alpha(\alpha) = 1$ and $f_\alpha(\tau\alpha) = f_\alpha(\tau^2\alpha) = 0$, and set $h_\alpha = f_\alpha \circ \Phi$.

Theorem 10.4 (Smirnov). *For $\alpha = 1, \tau, \tau^2$, H_α^δ converges uniformly on D to h_α as $\delta \to 0$.*

It follows, in particular, by taking $z \in \partial D$, that the asymptotic crossing probabilities for this percolation model are indeed conformally invariant and are given by Cardy's formula.

Before sketching the proof, we will describe a variant of the Cauchy–Riemann equations and of conjugate harmonic functions, associated with the angle $2\pi/3$. For $\alpha = 1, \tau, \tau^2$, and f analytic, set

$$f_\alpha = \mathrm{Re}(f/\alpha).$$

Then f_α is harmonic and we can recover f by

$$\alpha f = f_\alpha + \frac{i}{\sqrt{3}}(f_{\alpha\tau} - f_{\alpha\tau^2}).$$

Also, for any $\eta \in \mathbb{C}$, the directional derivatives satisfy

$$\nabla_\eta f_\alpha(z) = \frac{\partial}{\partial \varepsilon}\,|_{\varepsilon=0}\,\mathrm{Re}\left(\frac{f(z+\varepsilon\eta)}{\alpha}\right) = Re\left(\frac{f'(z)\eta}{\alpha}\right) = \nabla_{\tau\eta} f_{\tau\alpha}(z).$$

These are the $2\pi/3$-*Cauchy–Riemann equations*, and $(f_1, f_\tau, f_{\tau^2})$ is the *harmonic triple* of f.

Conversely, if we are given C^1 functions f_1, f_τ, f_{τ^2} such that, for $\alpha \in \{1, \tau, \tau^2\}$, for all η,

$$\nabla_\eta f_\alpha(z) = \nabla_{\tau\eta} f_{\tau\alpha}(z),$$

then f, defined by

$$f = f_1 + \frac{i}{\sqrt{3}}(f_\tau - f_{\tau^2}),$$

is analytic and $f_\alpha = \mathrm{Re}(f/\alpha)$ for all α.

Sketch proof of Theorem 10.4. For z the centre of a lattice triangle in D and η a vector from z to one of the three neighbouring triangle centres, for $\alpha \in \{1, \tau, \tau^2\}$, the events $Q = Q_\alpha(z+\eta) \setminus Q_\alpha(z)$ and $\tilde{Q} = Q_{\tau\alpha}(z+\tau\eta) \setminus Q_{\tau\alpha}(z)$ have the same probability. To see this, label the vertices of the triangle at z by X, Y, Z, where X is opposite to η and we move anticlockwise around the triangle. Note that Q is the event that there exist disjoint black paths from Y to $a(\alpha\tau^2)a(\alpha)$ and from Z to $a(\alpha)a(\tau\alpha)$ and also a white path from X to $a(\alpha\tau)a(\alpha\tau^2)$. On the other hand, $\tilde{Q}$ is a similar event but where the path from Y must be white, and that from X must be black. To see that $\mathbb{P}(Q) = \mathbb{P}(\tilde{Q})$, explore the lattice from $a(\alpha)$ just as far as is needed to find suitable black paths (for Q) from Y and Z. Supposing this done, the conditional probability of the required white path from X is the same as if we required it to be black (and disjoint from the other paths). Hence Q and $\tilde{Q}$ both have the same probability as the event of three disjoint black paths to the required boundary segments.

Set $P_\alpha(z,\eta) = \mathbb{P}(Q)$. We have shown that

$$P_\alpha(z,\eta) = P_{\tau\alpha}(z, \tau\alpha). \tag{9}$$

This is a discrete version of the $2\pi/3$-Cauchy–Riemann equations for the triple $(H_1, H_\tau, H_{\tau^2})$. The rest of the proof is analytic. We accept here without proof the following results

Lemma 10.5 (Hölder estimate)**.** *There are constants $\varepsilon > 0$ and $C < \infty$, depending only on $(D, a(1), a(\tau), a(\tau^2))$, such that*

$$|H_\alpha(z) - H_\alpha(z')| \leqslant C(|z-z'| \wedge \delta)^\varepsilon.$$

Also, $H_\alpha(a(\alpha)) \to 1$ as $\delta \to 0$.

The proof uses a a classical method for regularity estimates in percolation due to Russo, Seymour and Welsh.

Lemma 10.6. *For any equilateral triangular contour* Γ*, of side length* ℓ*, interpolating neighbouring centres of lattice triangles, define the discrete contour integral*

$$\int_\Gamma^\delta H(z)dz = \delta \sum_{z\in A_1} H(z) + \delta\tau \sum_{z\in A_\tau} H(z) + \delta\tau^2 \sum_{z\in A_{\tau^2}} H(z),$$

where A_α *is the set of centres along the side parallel to* α*. (Make some convention at the corners.) Then*

$$\int_\Gamma^\delta H_\alpha(z)dz = \frac{1}{\tau}\int_\Gamma^\delta H_{\alpha\tau}(z)dz + O(\ell\delta^\varepsilon).$$

The proof is an elementary, if complicated, resummation argument, using the identity

$$H_\alpha(z+\eta) - H_\alpha(z) = P_\alpha(z,\eta) - P_\alpha(z+\eta,-\eta)$$

and, from the preceding lemma, the estimate $P_\alpha(z,\eta) \leqslant C\delta^\varepsilon$ for some stray terms.

The Hölder estimate implies that every sequence $\delta_n \downarrow 0$ contains a subsequence δ_{n_k} such that $H_\alpha^{\delta_{n_k}}$ converges uniformly on D for all α, and any such subsequential limits, h_α say, must have boundary values $h_\alpha(a(\alpha)) = 1$ and $h_\alpha(z) = 0$ on $a(\alpha\tau)a(\alpha\tau^2)$. Moreover, by Lemma 10.6, we must have

$$\int_\Gamma h_\alpha(z)dz = \frac{1}{\tau}\int_\Gamma h_{\alpha\tau}(z)dz.$$

Set $h = h_1 + (i/\sqrt{3})(h_\tau - h_{\tau^2})$, then

$$\int_\Gamma h(z)dz = 0$$

for all Γ, so h is analytic by Morera's theorem, and $h_\alpha = \mathrm{Re}(h/\alpha)$ is harmonic for all α. Hence we obtain

$$\nabla_\eta h_\alpha = \nabla_{\tau\alpha} h_{\tau\alpha}.$$

(This can be considered as the limiting form of the key observation on the discrete model (9), but the limit has not been justified directly.) This relation implies that the directional derivatives of h_1 on $a(\tau^2)a(1)$ and $a(\tau)a(1)$ at an angle τ to the tangent are zero. Thus we have a (conformally-invariant) Dirichlet-Neumann problem for h_1. In the case $D = \Delta$, the affine function f_1 is obviously a solution, and moreover it is the only solution. Hence the functions $H_1^\delta, H_\tau^\delta, H_{\tau^2}^\delta$ each have exactly one uniform limit point as $\delta \to 0$, given by h_1, h_τ, h_{τ^2} respectively, as required. □

11 Further reading

Most of the content of these notes, and far more besides, can be found in Lawler's book [1]. Versions of Schramm's paper [4], which introduced SLE, and of some lecture notes by Werner [5], [6] are available at `uk.arxiv.org`.

References

[1] Gregory F. Lawler. *Conformally invariant processes in the plane*, volume 114 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.

[2] M. H. A. Newman. *Elements of the topology of plane sets of points*. Dover Publications Inc., New York, 1992.

[3] Ch. Pommerenke. *Boundary behaviour of conformal maps*, volume 299 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[4] Oded Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118:221–288, 2000.

[5] Wendelin Werner. Random planar curves and Schramm-Loewner evolutions. In *Lectures on probability theory and statistics*, volume 1840 of *Lecture Notes in Math.*, pages 107–195. Springer, Berlin, 2004.

[6] Wendelin Werner. Conformal restriction and related questions. *Probab. Surv.*, 2:145–190 (electronic), 2005.

Stochastic Loewner Evolutions: Examples Sheet 1

1. Suppose that f is an analytic function, defined on a neighbourhood of 0, with $f(0) = f'(0) = 0$ and $f''(0) = 1$. By considering the values of $\arg f$ round a small circle centred at 0, show that f is not a bijection. Deduce that, for any open set D in $\mathbb{C}$ and any analytic function Φ defined and one-to-one on D, we must have: (i) $\Phi(D)$ is an open set, (ii) Φ is a conformal map, (iii) the inverse map $\Phi^{-1} : \Phi(D) \to D$ is analytic.

2. Suppose that σ is a conformal isomorphism $\mathbb{H} \to \mathbb{H}$ fixing 0 and ∞. Show that there exists $\lambda \in (0, \infty$ such that $\sigma(z) = \lambda z$ for all z.

3. Let D be a simply connected domain in $\mathbb{C}$ and let $\Phi : D \to \mathbb{D}$ be a conformal isomorphism. Define a metric d on D by $d(z, w) = |\Phi(z) - \Phi(w)|$. Say that a sequence $(z_n)_{n\geqslant 1}$ in D is Φ-Cauchy if it is Cauchy for the metric d. Show that this property is in fact independent of the choice of Φ.

4. Use Itô's formula for C^2 functions to show that, for a proper complex domain D, if B is a complex Brownian motion starting from $z \in D$, and if u is a continuous function on $\bar{D}$ which is harmonic in D, then

$$u(z) = \mathbb{E}[u(B_{T_D})],$$

where $T_D = \inf\{t \geqslant 0 : B_t \notin D\}$.

5. Let D be a simply connected proper complex domain, whose boundary is a Jordan curve (a simple closed curve) ∂D. Let $z \in D$ and let $b_1, b_2, b_3 \in \partial D$ be distinct and positively ordered. Show that the conformal isomorphism provided by the Riemann mapping theorem $\Phi : D \to \mathbb{D}$ can be chosen to satisfy any one of the following three conditions, each of which then determines Φ uniquely:

(i) $\Phi(z) = 0$ and $\Phi'(z) > 0$;

(ii) $\Phi(z) = 0$ and $\Phi(b_1) = 1$;

(iii) $\Phi(b_1) = 1, \Phi(b_2) = i$ and $\Phi(b_3) = -1$.

6. Use the conformal invariance of harmonic measure to deduce from the case $z = 0$ the following formula for the hitting density of Brownian motion on the unit circle, starting from general $z \in \mathbb{D}$,

$$h_\Delta(z, t) = \frac{1}{2\pi}\frac{1 - |z|^2}{|e^{it} - z|^2}, \quad 0 \leqslant t < 2\pi.$$

Find an analogous formula for the case $|z| > 1$.

7. Show that there is only one conformal automorphism f of the upper half-plane $\mathbb{H}$ such that $f(z) - z \to 0$ as $z \to \infty$. Prove the uniqueness assertion of Proposition 4.5: *for a compact $\mathbb{H}$-hull K, there is at most one conformal isomorphism $g_K : \mathbb{H} \setminus K \to \mathbb{H}$ such that $g_K(z) - z \to 0$ as $z \to \infty$.*

Verify also that, if Φ is defined in a neighbourhood of 0 in $\mathbb{C}$, with

$$\Phi(z) = z + bz^2 + cz^3 + O(|z|^4)$$

as $z \to 0$, and if $g(z) = -1/\Phi(-1/z) - b$, then, as $z \to \infty$,

$$g(z) = z + \frac{b^2 - c}{z} + O(|z|^{-2}).$$

8. Show that $\operatorname{hcap}(K) \leqslant \operatorname{rad}(K)^2$ for all compact $\mathbb{H}$-hulls K. For which K does equality hold? Show also that there is a constant $C < \infty$ such that, for all compact $\mathbb{H}$-hulls K, we have

$$|z - g_K(z)| \leqslant C \operatorname{rad}(K), \quad z \in \mathbb{H} \setminus K$$

and, for all $\xi \in \bar{K} \cap \mathbb{R}$ and all $z \in \mathbb{H}$ with $|z - \xi| \geqslant 4 \operatorname{rad}(K)$,

$$\left| g_K(z) - z - \frac{\operatorname{hcap}(K)}{z - \xi} \right| \leqslant \frac{2C \operatorname{rad}(K) \operatorname{hcap}(K)}{|z - \xi|^2}.$$

9. Let K be a compact $\mathbb{H}$-hull. Show that

$$\operatorname{hcap}(K) = \lim_{y \to \infty} y \mathbb{E}_{iy}(\operatorname{Im}(B_T)),$$

where B is a complex Brownian motion starting from iy and $T = \inf\{t \geqslant 0 : B_t \notin \mathbb{H} \setminus K\}$.

10. Let $\gamma : [0, \infty) \to \bar{\mathbb{H}}$ be a simple path with $\gamma(0) = 0$, $\gamma_t \in \mathbb{H}$ for all $t > 0$ and $\operatorname{Im}(\gamma_t) \to \infty$ as $t \to \infty$. Set $K_t = \{\gamma_s : 0 < s \leqslant t\}$. Show that $(K_t)_{t \geqslant 0}$ is a strictly increasing family of compact $\mathbb{H}$-hulls, with $\operatorname{hcap}(K_t) \to \infty$ as $t \to \infty$, and having the local growth property.

11. Show that, for the solution flow $(g_t)_{t \geqslant 0}$ of the Loewner differential equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \xi_t}, \quad g_0(z) = z,$$

we have, as $z \to \infty$,

$$z(g_t(z) - z) \to 2t.$$

12. Let $(K_t)_{t \geqslant 0}$ be a family of compact $\mathbb{H}$-hulls having the local growth property. Show that the associated Loewner transform $(\xi_t)_{t \geqslant 0}$ is continuous.

13. Consider the family of compact $\mathbb{H}$-hulls $(K_t)_{t \geqslant 0}$ generated by the path

$$\gamma_t = \begin{cases} \frac{it}{1-t+it}, & t \leqslant 1, \\ 1 + i(t-1), & t > 1. \end{cases}$$

Does $(K_t)_{t \geqslant 0}$ have the local growth property? Justify your answer.

14. Let γ be an $SLE(\kappa)$. Define for $r > 0$ and $s \geqslant 0$

$$(\sigma_r \gamma)_t = r^{-1} \gamma_{r^2 t}, \quad (\theta_s \gamma)_t = g_s(\gamma_{s+t}) - \xi_s.$$

Express the Loewner tranforms of $\sigma_r \gamma$ and $\theta_s \gamma$ in terms of the Loewner transform of γ and hence show that both $\sigma_r \gamma$ and $\theta_s \gamma$ are also $SLE(\kappa)$.

15. Let $(X_t)_{t<\zeta}$ be the maximal solution to the Bessel stochastic differential equation

$$dX_t = dB_t + \frac{a}{X_t}dt, \quad X_0 = x,$$

where B is a Brownian motion. Fix $r > 0$ and set $Y_t = r^{-1}X_{r^2 t}$. Show that

$$dY_t = dW_t + \frac{a}{Y_t}dt,$$

for some Brownian motion W.

Suppose that $a \in (0, 1/2)$ so that $\zeta < \infty$, almost surely. Define for $r \in (0, x]$,

$$T(r) = \inf\{t \geqslant 0 : X_t = r\}.$$

Show that the random variables

$$A_n = \int_{T(2^{-n+1}x)}^{T(2^{-n}x)} \frac{1}{X_t^2}dt, \quad n \geqslant 1,$$

are independent and identically distributed.

Suppose now that $a \in (1/2, \infty)$. We know that $\zeta = \infty$ almost surely. Show that $\inf_{t\geqslant 0} X_t > 0$ almost surely.

Finally, consider the case $a = 1/2$. Show that $\zeta = \infty$ but $\inf_{t\geqslant 0} X_t = 0$ almost surely.

16. Show that, if $\Phi : D \to D'$ is a conformal isomorphism of $D, D' \in \mathcal{D}$ (that is, a conformal isomorphism of the underlying domains U, U', taking the two marked boundary points z_0, z_1 to their counterparts z_0', z_1'), then we can define a $\mathcal{C}_D/\mathcal{C}_{D'}$-measurable map $\Phi^* : C_D \to C_{D'}$ by

$$\Phi^*([\gamma]) = [\Phi \circ \gamma], \quad \gamma \in P_D.$$

Show also that, if μ is a scale-invariant measure on $(C, \mathcal{C})$, then there is a unique conformally invariant family of measures $(\mu_D : D \in \mathcal{D})$, with μ_D on $(C_D, \mathcal{C}_D)$, such that $\mu_{(\mathbb{H},0,\infty)} = \mu$.

17. Let $\Phi : N \to \mathbb{N}^*$ be a conformal isomorphism of simply connected planar domains. Suppose that $N \subseteq \mathbb{H}$ and that there exists an open interval I of the real axis with the following property: for all $x \in I$, there exists $\varepsilon > 0$ such that $z \in \mathbb{H}, |z - x| < \varepsilon$ implies $z \in N$. Show that Φ extends continuously to $N \cup I$.

Suppose now that $N^* \subseteq \mathbb{H}$ and that $I^* = \Phi(I)$ is an interval of the real axis. Show that, for any $x \in I$, Φ has an analytic extension to a neighbourhood of x in $\mathbb{C}$, with all derivatives real at x.

Stochastic Loewner Evolutions: Examples Sheet 2

1. Let $(X_t)_{t<\zeta}$ be the maximal solution to the Bessel stochastic differential equation

$$dX_t = dB_t + \frac{a}{X_t}dt, \quad X_0 = x,$$

where B is a Brownian motion. Fix $r > 0$ and set $Y_t = r^{-1}X_{r^2t}$. Show that

$$dY_t = dW_t + \frac{a}{Y_t}dt,$$

for some Brownian motion W.

Suppose that $a \in (0, 1/2)$ so that $\zeta < \infty$, almost surely. Define for $r \in (0, x]$,

$$T(r) = \inf\{t \geqslant 0 : X_t = r\}.$$

Show that the random variables

$$A_n = \int_{T(2^{-n+1}x)}^{T(2^{-n}x)} \frac{1}{X_t^2}dt, \quad n \geqslant 1,$$

are independent and identically distributed.

Suppose now that $a \in (1/2, \infty)$. We know that $\zeta = \infty$ almost surely. Show that $\inf_{t\geqslant 0} X_t > 0$ almost surely.

Finally, consider the case $a = 1/2$. Show that $\zeta = \infty$ but $\inf_{t\geqslant 0} X_t = 0$ almost surely.

2. Show that, if $\Phi : D \to D'$ is a conformal isomorphism of $D, D' \in \mathcal{D}$ (that is, a conformal isomorphism of the underlying domains U, U', taking the two marked boundary points z_0, z_1 to their counterparts z_0', z_1'), then we can define a $\mathcal{C}_D/\mathcal{C}_{D'}$-measurable map $\Phi^* : C_D \to C_{D'}$ by

$$\Phi^*([\gamma]) = [\Phi \circ \gamma], \quad \gamma \in P_D.$$

Show also that, if μ is a scale-invariant measure on $(C, \mathcal{C})$, then there is a unique conformally invariant family of measures $(\mu_D : D \in \mathcal{D})$, with μ_D on $(C_D, \mathcal{C}_D)$, such that $\mu_{(\mathbb{H},0,\infty)} = \mu$.

3. Let $\Phi : N \to N^*$ be a conformal isomorphism of simply connected planar domains. Suppose that $N \subseteq \mathbb{H}$ and that there exists an open interval I of the real axis with the following property: for all $x \in I$, there exists $\varepsilon > 0$ such that $z \in \mathbb{H}, |z - x| < \varepsilon$ implies $z \in N$. Show that Φ extends continuously to $N \cup I$.

Suppose now that $N^* \subseteq \mathbb{H}$ and that $I^* = \Phi(I)$ is an interval of the real axis. Show that, for any $x \in I$, Φ has an analytic extension to a neighbourhood of x in $\mathbb{C}$, with all derivatives real at x.

4. Suppose that a function Φ is analytic near ξ, with $\Phi'(\xi) > 0$. Show that, as $z \to \xi$,

$$\frac{2\Phi'(\xi)^2}{\Phi(z) - \Phi(\xi)} - \Phi'(z)\frac{2}{z - \xi} \to -3\Phi''(\xi)$$

and

$$2\left(-\frac{\Phi'(\xi)^2\Phi'(z)}{(\Phi(z)-\Phi(\xi))^2}+\frac{\Phi'(z)}{(z-\xi)^2}-\frac{\Phi''(z)}{z-\xi}\right)\to\frac{1}{2}\frac{\Phi''(\xi)^2}{\Phi'(\xi)}-\frac{4}{3}\Phi'''(\xi).$$

It is worth reducing to the case $\xi=\Phi(\xi)=0$ and $\Phi'(\xi)=1$ (how?) before starting to compute.

THE REMAINING QUESTIONS FORMED LAST YEAR'S EXAMINATION

5. Let $(K_t)_{t\geqslant 0}$ be a strictly increasing family of *compact* $\mathbb{H}$-*hulls* with $\mathrm{hcap}(K_t)=2t$ for all t and with the *local growth property*, having *Loewner transform* $(\xi_t)_{t\geqslant 0}$. Explain all the italicized terms in the preceding sentence and state how one can obtain $(\xi_t)_{t\geqslant 0}$ from $(K_t)_{t\geqslant 0}$. Discuss briefly also how to reconstruct $(K_t)_{t\geqslant 0}$ from $(\xi_t)_{t\geqslant 0}$.

Let $\kappa\in[0,\infty)$. What is meant by saying that a continuous process $(\gamma_t)_{t\geqslant 0}$ is (chordal, half-plane) $SLE(\kappa)$?

Show carefully that $SLE(\kappa)$ is scale-invariant.

Explain how it is possible to define in a consistent way $SLE(\kappa)$ in any simply connected Jordan domain D from one given boundary point z_0 to another z_1. (You may assume that $\gamma_t\to\infty$ as $t\to\infty$ almost surely.)

6. Let γ be an $SLE(\kappa)$, with $\kappa\in(0,4)$. Show that, almost surely, γ is a simple curve and $\gamma_t\to\infty$ as $t\to\infty$. You may use any facts about Bessel processes you wish, without proof, provided that these are clearly stated.

7. Fix $a\in(0,1/2)$. For $x,y\in(0,\infty)$, define processes X and Y by the stochastic differential equations

$$dX_t=dB_t+\frac{a}{X_t},\quad dY_t=-dB_t+\frac{a}{Y_t},\quad X_0=x,\quad Y_0=y,$$

where B is a Brownian motion, and we consider X as defined up to the first time ζ that it hits 0, and similarly Y as defined up to the first time τ that it hits 0. Show that $\zeta<\infty$, almost surely.

Show moreover that

$$\mathbb{P}(\zeta<\tau)=c\int_0^{y/(x+y)}\frac{du}{u^{2a}(1-u)^{2a}},$$

for some constant c independent of x and y.

Discuss briefly how this probability can be interpreted, for a suitable value of a, which you should specify, as a crossing probability for the continuum limit of critical planar percolation.

8. Explain what is meant by a *filling* of a simply connected complex domain, with two chosen points of the conformal boundary $D=(U,z_0,z_1)$.

Suppose that $(\mu_D:D\in\mathcal{D})$ is a conformally invariant family of probability measures, indexed by the set $\mathcal{D}$ of all such D, where μ_D is a measure on fillings in D. What does it mean to say that $(\mu_D:D\in\mathcal{D})$ has the restriction property? Express this property in terms of the single measure μ corresponding to $D=(\mathbb{H},0,\infty)$.

Let γ be an $SLE(8/3)$ and let E be a Brownian excursion in $\mathbb{H}$, from 0 to ∞. Thus $E = (B, |W|)$ with B and W independent Brownian motions, starting from 0, in $\mathbb{R}$ and $\mathbb{R}^3$ respectively. For D a filling of $(\mathbb{H}, 0, \infty)$, it is known that

$$\mathbb{P}(\gamma_t \in D \text{ for all } t \geqslant 0) = (\Phi'_D(0))^{5/8}, \quad \mathbb{P}(\mathbb{E}_t \in D \text{ for all } t \geqslant 0) = \Phi'_D(0),$$

where Φ_D is the conformal isomorphism $D \to \mathbb{H}$ with $\Phi'_D(\infty) = 1$. [*You are not expected to prove these facts in this question.*] Deduce that the fillings $\bar{\gamma}$ and $\bar{E}$, generated by γ and E respectively, both have the restriction property.

Show further that the law of the filling $\bar{\gamma}^{\otimes 8}$ generated by eight independent copies of γ coincides with the law of the filling $\bar{E}^{\otimes 5}$ generated by five independent copies of E.