STOCHASTIC FINANCIAL MODELS

Example Sheet 1

1. An investor with strictly concave utility function U seeks to maximize expected utility $\mathbb{E}(U(X))$ over a convex set $\mathcal{A}$ of available contingent claims X. Assume that $U(X)$ is integrable for all $X \in \mathcal{A}$. Show that there is at most one maximizing contingent claim.

2. (a) Show that the utility function of a risk-neutral investor is affine.

(b) Suppose that the utility function U of an investor has the following property: the investor is indifferent between any two contingent claims having the same mean and variance. Show that U must be quadratic.

3. Let X be an $N(\mu, \sigma^2)$ random variable. Show that, for all $\theta \in \mathbb{R}$ and all suitable functions f, the following identity holds

$$\mathbb{E}(f(X + \theta\sigma)) = \mathbb{E}\left(\exp\left\{\theta(X - \mu)/\sigma - \theta^2/2\right\} f(X)\right)$$

and deduce that

$$\mathbb{E}((X - \mu)f(X)) = \sigma^2 \mathbb{E}(f'(X)).$$

Show further that, for all $\alpha, \beta \in \mathbb{R}$,

$$\mathbb{E}(\Phi(\alpha X + \beta)) = \Phi\left(\frac{\alpha\mu + \beta}{\sqrt{1 + \alpha^2\sigma^2}}\right)$$

where Φ denotes the standard Gaussian distribution function.

4. Let (X, Y) be a bivariate normal random variable. Show that $Y - aX$ is independent of X for some constant a, to be determined. Deduce that, for suitable functions f,

$$\operatorname{cov}(f(X), Y) = \mathbb{E}(f'(X)) \operatorname{cov}(X, Y).$$

5. Let $U : \mathbb{R} \to \mathbb{R}$ be a concave function. Given a constant $\mu \in \mathbb{R}$ and zero-mean random variable Z, set

$$\phi(t) = \mathbb{E}(U(\mu + tZ)).$$

Show that ϕ is concave on $\mathbb{R}$ and is non-increasing on $[0, \infty)$. Show further that, if U is strictly concave and $\mathbb{P}(Z = 0) < 1$, then ϕ is also strictly concave and is decreasing on $[0, \infty)$.

6. Let $U : \mathbb{R} \to \mathbb{R}$ be a strictly concave increasing function. Define, for $\mu \in \mathbb{R}$ and $\sigma \in (0, \infty)$,

$$u(\mu, \sigma) = \int_{\mathbb{R}} U(x) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)} dx, \quad u(\mu, 0) = U(\mu).$$

Show that u is strictly concave on $\mathbb{R} \times [0, \infty)$, increasing in μ and decreasing in σ.

Hence show, for a Gaussian single-period asset price model (S_0, S_1), that an investor maximizing expected utility $\mathbb{E}(U(\theta.S_1))$, subject to given initial wealth $\theta.S_0$, will always choose a portfolio θ on the mean-variance-efficient frontier.

7. (a) Fix $\varepsilon > 0$ and consider an agent with 'utility function' $U(x) = x - \varepsilon x^2/2$. (While U is decreasing in x for $x > 1/\varepsilon$, so U is not a true utility function, nevertheless we can consider the pricing of contingent claims X by maximizing $\mathbb{E}(U(X))$.) Suppose that the agent has wealth x at time 0, and that he may invest any amount $\theta \in \mathbb{R}$ in a single stock having price $S_0 = 1$ at time 0 and a random price S_1 at time 1. His remaining wealth (or debt) is held in a bond whose value is constant. Set $Z = S_1 - S_0$ and define

$$u(x) = \sup_{\theta \in \mathbb{R}} \mathbb{E}(U(x + \theta Z)).$$

Find the unique optimal investment $\theta^*(x)$ such that $u(x) = \mathbb{E}(U(X^*))$ for $X^* = x + \theta^*(x)Z$, and show that $\mathbb{E}(U'(X^*)Z) = 0$.

(b) Assume from now on that x is chosen so that $u'(x) > 0$. The agent is offered the possibility to buy some multiple tY of another contingent claim Y. Show that, for small t, the agent's bid price $\pi(t)$ for tY is determined uniquely by

$$u(x - \pi(t), t) = u(x)$$

where

$$u(x, t) = \sup_{\theta \in \mathbb{R}} \mathbb{E}(U(x + tY + \theta Z)).$$

(c) Hence show that $\pi(t)$ is differentiable at $t = 0$, with

$$\dot{\pi}(0) = \frac{\mathbb{E}(U'(X^*)Y)}{\mathbb{E}(U'(X^*))}.$$

8. (a) Consider a single period model with d risky assets. For $n = 0, 1$, write $S_n = (S_n^1, \dots, S_n^d)$ for the vector of asset values at time n. We assume that S_0 is non-random and that $S_1 \sim N(\mu, V)$ for some $\mu \in \mathbb{R}^d$ and some invertible covariance matrix V. An agent, with C^2 concave utility function U, aims to maximize her expected utility $\mathbb{E}(U(\theta.S_1))$ at time 1 over portfolios $\theta \in \mathbb{R}^d$, subject to the constraint that $\theta.S_0 = w_0$, where w_0 denotes her wealth at time 0. Show that the optimal portfolio has the form

$$\theta^* = \frac{A\mu}{\gamma} + \frac{\gamma w_0 - S_0.(A\mu)}{\gamma S_0.(AS_0)} AS_0$$

where

$$A = V^{-1}, \quad \gamma = -\frac{\mathbb{E}(U''(X_1^*))}{\mathbb{E}(U'(X_1^*))}, \quad X_1^* = \theta^*.S_1.$$

(This has the same form as the optimal portfolio for CARA utility function with coefficient of absolute risk aversion γ.) [*Hint: Question 4 may be useful.*]

(b) Suppose now that we add a riskless asset, having value $S_0^0 = 1$ at time 0 and having value $S_1^0 = 1 + r$ at time 1. Write $\bar{S}_n$ for the augmented vector of asset values $(S_n^0, S_n^1, \dots, S_n^d)$. The agent now aims to maximize $\mathbb{E}(U(\bar{\theta}.\bar{S}_1))$ over $\bar{\theta} \in \mathbb{R}^{d+1}$, subject to the constraint $\bar{\theta}.\bar{S}_0 = w_0$. Show that, for an analogous interpretation of γ, the optimal portfolio is again of the form associated with the CARA utility function with coefficient of absolute risk aversion γ.

9. Consider a single period model with one riskless asset and d risky assets. Assume that the riskless asset has value 1 at time 0 and has value $1 + r$ at time 1. Write S_0 and S_1 for the vectors of risky asset prices at time 0 and time 1 respectively. A market for the risky assets is formed of K agents, who hold between them α_i units of asset i before trading begins. Suppose, for $k = 1, \ldots, K$, that agent k has CARA utility function

$$U_k(x) = -e^{-\gamma(k)x}$$

with $\gamma(k) > 0$ for all k. At time 0, the agents share a common belief that $S_1 \sim N(\mu, V)$ for some $\mu \in \mathbb{R}^d$ and some invertible covariance matrix V. Determine the equilibrium price vector S_0 which clears the market, that is, which requires no net transfer of any risky asset into or out of the market. Show further that if $\alpha_i = 0$ for some i, then no agent takes a position in asset i after trading, and, for all j, the price of asset j at time 0 is unaffected by any correlation in prices of assets i and j at time 1.