Stochastic Calculus and Applications Lent 2007

Example Sheet 4

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Relations with partial differential equations

1. Let $\sigma_1, \dots, \sigma_m$ and b be bounded Lipschitz vector fields on $\mathbb{R}^d$. Suppose that $(X_t)_{t \geq 0}$ is a solution of the stochastic differential equation in $\mathbb{R}^d$

$$dX_t = \sum_{i=1}^m \sigma_i(X_t) dB_t^i + b(X_t)dt,$$

starting from x_0, where $B = (B^1, \dots, B^m)$ is a Brownian motion in $\mathbb{R}^m$. Assume that

$$\inf_{x,\xi \in \mathbb{R}^d} \sum_{i=1}^m \langle \xi, \sigma_i(x) \rangle^2 > 0.$$

Show that, for $n \geq 1$ and $0 = t_0 < t_1 < \cdots < t_n$, the $(\mathbb{R}^d)^n$-valued random variable $(X_{t_1}, \dots, X_{t_n})$ has a density with respect to the n-fold product of Lebesgue measure on $\mathbb{R}^d$ of the form

$$f(x_1, \dots, x_n) = \prod_{i=0}^{n-1} p(t_{i+1} - t_i, x_i, x_{i+1})$$

and explain how the function $p(t, x, y)$ may be obtained by solving a suitable partial differential equation. You are not expected to prove any results needed from the theory of partial differential equations.

2. Let $a, b, c : \mathbb{R} \to \mathbb{R}$ be bounded with bounded derivatives and with $a > 0$. Suppose that $u \in C_b^{1,2}(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$ solves the Cauchy problem

$$\frac{\partial u}{\partial t} = Lu \text{ in } \mathbb{R}^+ \times \mathbb{R}, \quad u(0, \cdot) = f \text{ on } \mathbb{R}$$

where

$$Lu = \frac{1}{2} a(x) u'' + b(x) u' + c(x) u.$$

Set $\sigma = \sqrt{a}$ and define X by the SDE

$$dX_t = \sigma(X_t) dB_t + b(X_t) dt, \quad X_0 = x.$$

Define also $E_t = \exp\{\int_0^t c(X_s) ds\}$. Show that

$$u(t, x) = \mathbb{E}_x\big(E_t f(X_t)\big).$$

3. Let $b : \mathbb{R}^d \to \mathbb{R}^d$ be Lipschitz and let $x_0 \in \mathbb{R}^d$. Consider for each $\varepsilon > 0$ the diffusion process X^ε in $\mathbb{R}^d$, starting from x_0 and having generator

$$L^\varepsilon = (1/2)\varepsilon^2 \Delta + b(x).\nabla.$$

Show that, for all $t \geq 0$,

$$\sup_{s \leq t} |X_s^\varepsilon - x_s| \to 0$$

in probability as $\varepsilon \to 0$, where $(x_t)_{t \geq 0}$ is given by the differential equation $\dot{x}_t = b(x_t)$, starting from x_0.

Jump processes

4. Let T be an exponential random variable of parameter λ and let W be any integrable random variable. Assume that, for some measurable function h, we have $\mathbb{E}(W|T) = h(T)$ almost surely. Define
$$M_t = W1_{\{T \leq t\}} - \lambda \int_0^{T \wedge t} h(s)ds$$
and
$$\mathcal{F}_t = \sigma(\{T > t\}, \{T \leq t, W \leq w\} : w \in \mathbb{R}).$$
Show that M is an $(\mathcal{F}_t)$-martingale.

5. For $x = (x^1, x^2) \in \mathbb{R}^2 \setminus \{0\}$ set $n(x) = (-x^2, x^1)/|x|$ and consider the Markov jump process X in $\mathbb{R}^2$ having generating kernel
$$q(x, A) = \begin{cases} |x|, & \text{if } x + n(x) \in A, \\ 0, & \text{if } x + n(x) \notin A, \end{cases}$$
and starting from $X_0 = (a, 0)$, where $a > 0$. Show that (a) for all a,
$$\mathbb{P}(|X_t| \to \infty \text{ as } t \to \infty) = 1,$$
and (b) for all $t > 0$,
$$\sup_{s \leq t} |a^{-1}X_s - (\cos s, \sin s)| \to 0$$
in probability as $a \to \infty$.

6. In the playground of a large school there is a craze for paper-scissors-stone. This is game for two players, who simultaneously declare themselves to be paper, scissors or stone by an agreed signal. The winner is decided as follows: scissors cuts paper, stone blunts scissors and paper smothers stone. The loser of any game switches to his opponent's choice next time, whilst the winner sticks with the same choice. Suppose that games take place as a Poisson process of rate λ, and that the pair of players is chosen randomly from the N children in the playground. Write down a Markov chain model for the proportions of the total number of children choosing paper, scissors and stone.

   Show that, if λ is taken to be a suitable function of N, then, in the limit as $N \to \infty$, the Markov chain is well approximated, in a sense you should make precise, by the system of differential equations:
$$\begin{aligned} \dot{x}_t^1 &= x_t^1(x_t^3 - x_t^2), \\ \dot{x}_t^2 &= x_t^2(x_t^1 - x_t^3), \\ \dot{x}_t^3 &= x_t^3(x_t^2 - x_t^1). \end{aligned}$$
   Show that $x_t^1 x_t^2 x_t^3$ is a constant and comment on this fact in the light of the long-time behaviour of the approximating Markov chains.