Probability and Measure 3

4.1 Let X be a random variable and let $1 \leq p < \infty$. Show that, if $X \in L^p(\mathbb{P})$, then $\mathbb{P}(|X| \geq \lambda) = O(\lambda^{-p})$ as $\lambda \to \infty$. Prove the identity

$$\mathbb{E}(|X|^p) = \int_0^\infty p\lambda^{p-1}\mathbb{P}(|X| \geq \lambda)d\lambda$$

and deduce that, for all $q > p$, if $\mathbb{P}(|X| \geq \lambda) = O(\lambda^{-q})$ as $\lambda \to \infty$, then $X \in L^p(\mathbb{P})$.

4.2 Give a simple proof of Schwarz' inequality $\|fg\|_1 \leq \|f\|_2\|g\|_2$ for measurable functions f and g.

4.3 Show that $\|XY\|_1 = \|X\|_1\|Y\|_1$ for independent random variables X and Y. Show further that, if X and Y are also integrable, then $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.

4.4 A *stepfunction* $f : \mathbb{R} \to \mathbb{R}$ is any finite linear combination of indicator functions of finite intervals. Show that the set of stepfunctions $\mathcal{I}$ is dense in $L^p(\mathbb{R})$ for all $p \in [1, \infty)$: that is, for all $f \in L^p(\mathbb{R})$ and all $\varepsilon > 0$ there exists $g \in \mathcal{I}$ such that $\|f - g\|_p < \varepsilon$. Deduce that the set of continuous functions of compact support is also dense in $L^p(\mathbb{R})$ for all $p \in [1, \infty)$.

4.5 Let $(X_n : n \in \mathbb{N})$ be an identically distributed sequence in $L^2(\mathbb{P})$. Show that $n\mathbb{P}(|X_1| > \varepsilon\sqrt{n}) \to 0$ as $n \to \infty$, for all $\varepsilon > 0$. Deduce that $n^{-1/2} \max_{k \leq n} |X_k| \to 0$ in probability.

5.1 Let $(E, \mathcal{E}, \mu)$ be a measure space and let $V_1 \leq V_2 \leq \dots$ be an increasing sequence of closed subspaces of $L^2 = L^2(E, \mathcal{E}, \mu)$ for $f \in L^2$, denote by f_n the orthogonal projection of f on V_n. Show that f_n converges in L^2.

5.2 Let $X = (X_1, \dots, X_n)$ be a random variable, with all components in $L^2(\mathbb{P})$. The covariance matrix $\text{var}(X) = (c_{ij} : 1 \leq i, j \leq n)$ of X is defined by $c_{ij} = \text{cov}(X_i, X_j)$. Show that $\text{var}(X)$ is a non-negative definite matrix.

6.1 Find a uniformly integrable sequence of random variables $(X_n : n \in \mathbb{N})$ such that both $X_n \to 0$ a.s. and $\mathbb{E}\big(\sup_n |X_n|\big) = \infty$.

6.3 Let $(X_n : n \in \mathbb{N})$ be an identically distributed sequence in $L^2(\mathbb{P})$. Show that

$$\mathbb{E}\big(\max_{k \leq n} |X_k|\big)/\sqrt{n} \to 0 \qquad \text{as} \quad n \to \infty.$$

7.1 Show that the Fourier transform of a finite Borel measure is a bounded continuous function.

7.2 Determine which of the following distributions have an integrable characteristic function: normal, binomial, Poisson, U[0,1].

7.3 Show that there do not exist independent identically distributed random variables X, Y such that $X - Y \sim U[-1, 1]$.

7.4 The Cauchy distribution has density function

$$f(x) = \frac{1}{\pi(1 + x^2)}, \quad x \in \mathbb{R}.$$

Show that the corresponding characteristic function is given by

$$\varphi(u) = e^{-|u|}.$$

Show also that, if $X_1, \ldots, X_n$ are independent Cauchy random variables, then the random variable $(X_1 + \cdots + X_n)/n$ is also Cauchy.

7.5 For a finite Borel measure μ on the line show that, if $\int |x|^k d\mu(x) < \infty$, then the Fourier transform $\hat{\mu}$ of μ has a kth continuous derivative, which at 0 is given by

$$\hat{\mu}^{(k)}(0) = i^k \int x^k d\mu(x).$$

7.6 (i) Show that for any real numbers a, b one has $\int_a^b e^{itx} dx \to 0$ as $|t| \to \infty$.

(ii) Show that, for any $f \in L^1(\mathbb{R})$, the Fourier transform

$$\hat{f}(t) = \int_{-\infty}^{\infty} e^{itx} f(x) dx$$

tends to 0 as $|t| \to \infty$. This is the *Riemann–Lebesgue Lemma.*