

Probability and Measure 1

1.1 Let E be a set and let \mathcal{S} be a set of σ -algebras on E . Define

$$\mathcal{E}^* = \{A \subseteq E : A \in \mathcal{E} \text{ for all } \mathcal{E} \in \mathcal{S}\}.$$

Show that \mathcal{E}^* is a σ -algebra on E . Show, on the other hand, by example, that the union of two σ -algebras on the same set need not be a σ -algebra.

1.2 Show that the following sets of subsets of \mathbb{R} all generate the same σ -algebra:

$$(a) \{(a, b) : a < b\}, \quad (b) \{(a, b] : a < b\}, \quad (c) \{(-\infty, b] : b \in \mathbb{R}\}.$$

1.3 Show that a countably additive set function on a ring is additive, increasing and countably subadditive.

1.4 Show that a π -system which is also a d -system is a σ -algebra.

1.5 Let μ be a finite-valued additive set function on a ring \mathcal{A} . Show that μ is countably additive if and only if the following condition holds: for any decreasing sequence $(A_n : n \in \mathbb{N})$ of sets in \mathcal{A} , with $\bigcap_n A_n = \emptyset$, we have $\mu(A_n) \rightarrow 0$.

1.6 Let (E, \mathcal{E}, μ) be a finite measure space. Show that, for any sequence of sets $(A_n : n \in \mathbb{N})$ in \mathcal{E} ,

$$\mu(\liminf A_n) \leq \liminf \mu(A_n) \leq \limsup \mu(A_n) \leq \mu(\limsup A_n).$$

Show that the first inequality remains true without the assumption that μ is finite, but the last inequality may be false.

1.7 Let $(A_n : n \in \mathbb{N})$ be a sequence of events in a probability space. Show that the events A_n are independent if and only if the σ -algebras $\sigma(A_n) = \{\emptyset, A_n, A_n^c, \Omega\}$ are independent.

1.8 Let B be a Borel subset of the interval $[0, 1]$. Show that for every $\varepsilon > 0$, there exists a finite union of disjoint intervals $A = (a_1, b_1] \cup \dots \cup (a_n, b_n]$ such that the Lebesgue measure of $A \Delta B (= (A^c \cap B) \cup (A \cap B^c))$ is less than ε . Show further that this remains true for every Borel set in \mathbb{R} of finite Lebesgue measure.

1.9 Let (E, \mathcal{E}, μ) be a measure space. Call a subset $N \subseteq E$ *null* if $N \subseteq B$ for some $B \in \mathcal{E}$ with $\mu(B) = 0$. Write \mathcal{N} for the set of null sets. Prove that the set of subsets $\mathcal{E}^\mu = \{A \cup N : A \in \mathcal{E}, N \in \mathcal{N}\}$ is a σ -algebra and show that μ has a well-defined and countably additive extension to \mathcal{E}^μ given by $\mu(A \cup N) = \mu(A)$. We call \mathcal{E}^μ the *completion of \mathcal{E} with respect to μ* .

2.1 Let $(f_n : n \in \mathbb{N})$ be a sequence of measurable functions on a measurable space (E, \mathcal{E}) . Show that the following functions are also measurable: $f_1 + f_2$, $f_1 f_2$, $\inf_n f_n$, $\sup_n f_n$, $\liminf_n f_n$, $\limsup_n f_n$. Show also that $\{x \in E : f_n(x) \text{ converges as } n \rightarrow \infty\} \in \mathcal{E}$.

2.2 Let (E, \mathcal{E}) and (G, \mathcal{G}) be measurable spaces, let μ be a measure on \mathcal{E} , and let $f : E \rightarrow G$ be a measurable function. Show that we can define a measure ν on \mathcal{G} by setting $\nu(A) = \mu(f^{-1}(A))$ for each $A \in \mathcal{G}$.

2.3 Show that the following condition implies that random variables X and Y are independent: $\mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y)$ for all $x, y \in \mathbb{R}$.

2.4 Let $(A_n : n \in \mathbb{N})$ be a sequence of events, with $\mathbb{P}(A_n) = 1/n^2$ for all n . Set $X_n = n^2 1_{A_n} - 1$ and consider the random variables $\bar{X}_n = (X_1 + \dots + X_n)/n$. Show that $\mathbb{E}(\bar{X}_n) = 0$ for all n , but that $\bar{X}_n \rightarrow -1$ almost surely as $n \rightarrow \infty$.

2.5 The zeta function is defined for $s > 1$ by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. Let X and Y be independent random variables with

$$\mathbb{P}(X = n) = \mathbb{P}(Y = n) = n^{-s}/\zeta(s).$$

Write A_n for the event that n divides X . Show that the events $(A_p : p \text{ prime})$ are independent and deduce Euler's formula

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right).$$

Show also that $\mathbb{P}(X \text{ is square-free}) = 1/\zeta(2s)$. Write H for the highest common factor of X and Y . Show finally that $\mathbb{P}(H = n) = n^{-2s}/\zeta(2s)$.

2.6 Let X_1, X_2, \dots be independent $N(0, 1)$ random variables. Prove that

$$\limsup_n (X_n/\sqrt{2 \log n}) = 1 \quad \text{a.s.}$$

2.7 Let C_n denote the n th approximation to the Cantor set C : thus $C_0 = [0, 1]$, $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, etc. and $C_n \downarrow C$ as $n \rightarrow \infty$. Denote by F_n the distribution function of a random variable uniformly distributed on C_n . Show that

- (i) C is uncountable and has Lebesgue measure 0,
- (ii) for all $x \in [0, 1]$, the limit $F(x) = \lim_{n \rightarrow \infty} F_n(x)$ exists,
- (iii) the function F is continuous on $[0, 1]$, with $F(0) = 0$ and $F(1) = 1$,
- (iv) for almost all $x \in [0, 1]$, F is differentiable at x with $F'(x) = 0$.

Hint: express F_{n+1} recursively in terms of F_n and use this relation to obtain a uniform estimate on $F_{n+1} - F_n$.