3. Integration

3.1. **Definition of the integral and basic properties.** Let $(E, \mathcal{E}, \mu)$ be a measure space. We shall define, where possible, for a measurable function $f : E \to [-\infty, \infty]$, the *integral* of f, to be denoted

$$\mu(f) = \int_E f d\mu = \int_E f(x)\mu(dx).$$

For a random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the integral is usually called instead the *expectation* of X and written $\mathbb{E}(X)$.

A *simple* function is one of the form

$$f = \sum_{k=1}^{m} a_k 1_{A_k}$$

where $0 \leq a_k < \infty$ and $A_k \in \mathcal{E}$ for all k, and where $m \in \mathbb{N}$. For simple functions f, we define

$$\mu(f) = \sum_{k=1}^{m} a_k \mu(A_k),$$

where we adopt the convention $0.\infty = 0$. Although the representation of f is not unique, it is straightforward to check that $\mu(f)$ is well defined and, for simple functions f, g and constants $\alpha, \beta \geq 0$, we have

(a) $\mu(\alpha f + \beta g) = \alpha\mu(f) + \beta\mu(g)$,
(b) $f \leq g$ implies $\mu(f) \leq \mu(g)$,
(c) $f = 0$ a.e. if and only if $\mu(f) = 0$.

In particular, for simple functions f, we have

$$\mu(f) = \sup\{\mu(g) : g \text{ simple}, \ g \leq f\}.$$

We define the integral $\mu(f)$ of a non-negative measurable function f by

$$\mu(f) = \sup\{\mu(g) : g \text{ simple}, \ g \leq f\}.$$

We have seen that this is consistent with our definition for simple functions. Note that, for all non-negative measurable functions f, g with $f \leq g$, we have $\mu(f) \leq \mu(g)$. For any measurable function f, set $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$. Then $f = f^+ - f^-$ and $|f| = f^+ + f^-$. If $\mu(|f|) < \infty$, then we say that f is *integrable* and define

$$\mu(f) = \mu(f^+) - \mu(f^-).$$

Note that $|\mu(f)| \leq \mu(|f|)$ for all integrable functions f. We sometimes define the integral $\mu(f)$ by the same formula, even when f is not integrable, but when either $\mu(f^-)$ or $\mu(f^+)$ is finite. In such cases the integral take the value ∞ or $-\infty$.

Here is the key result for the theory of integration. For $x \in [0, \infty]$ and a sequence $(x_n : n \in \mathbb{N})$ in $[0, \infty]$, we write $x_n \uparrow x$ to mean that $x_n \leq x_{n+1}$ for all n and $x_n \to x$

as $n \to \infty$. For a non-negative function f on E and a sequence of such functions $(f_n : n \in \mathbb{N})$, we write $f_n \uparrow f$ to mean that $f_n(x) \uparrow f(x)$ for all $x \in E$.

Theorem 3.1.1 (Monotone convergence). *Let f be a non-negative measurable function and let $(f_n : n \in \mathbb{N})$ be a sequence of such functions. Suppose that $f_n \uparrow f$. Then $\mu(f_n) \uparrow \mu(f)$.*

Proof. Case 1: $f_n = 1_{A_n}, f = 1_A$.
The result is a simple consequence of countable additivity.
Case 2: f_n *simple*, $f = 1_A$.
Fix $\varepsilon > 0$ and set $A_n = \{f_n > 1 - \varepsilon\}$. Then $A_n \uparrow A$ and

$$(1 - \varepsilon)1_{A_n} \leq f_n \leq 1_A$$

so

$$(1 - \varepsilon)\mu(A_n) \leq \mu(f_n) \leq \mu(A).$$

But $\mu(A_n) \uparrow \mu(A)$ by Case 1 and $\varepsilon > 0$ was arbitrary, so the result follows.

Case 3: f_n *simple,* f *simple.*
We can write f in the form

$$f = \sum_{k=1}^{m} a_k 1_{A_k}$$

with $a_k > 0$ for all k and the sets A_k disjoint. Then $f_n \uparrow f$ implies

$$a_k^{-1} 1_{A_k} f_n \uparrow 1_{A_k}$$

so, by Case 2,

$$\mu(f_n) = \sum_k \mu(1_{A_k} f_n) \uparrow \sum_k a_k \mu(A_k) = \mu(f).$$

Case 4: f_n *simple,* $f \geq 0$ *measurable.*
Let g be simple with $g \leq f$. Then $f_n \uparrow f$ implies $f_n \wedge g \uparrow g$ so, by Case 3,

$$\mu(f_n) \geq \mu(f_n \wedge g) \uparrow \mu(g).$$

Since g was arbitrary, the result follows.
Case 5: $f_n \geq 0$ *measurable,* $f \geq 0$ *measurable.*
Set $g_n = (2^{-n}\lfloor 2^n f_n \rfloor) \wedge n$ then g_n is simple and $g_n \leq f_n \leq f$, so

$$\mu(g_n) \leq \mu(f_n) \leq \mu(f).$$

But $f_n \uparrow f$ forces $g_n \uparrow f$, so $\mu(g_n) \uparrow \mu(f)$, by Case 4, and so $\mu(f_n) \uparrow \mu(f)$. □