2.3. **Random variables.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(E, \mathcal{E})$ be a measurable space. A measurable function $X : \Omega \to E$ is called a *random variable in* E. It has the interpretation of a quantity, or state, determined by chance. Where no space E is mentioned, it is assumed that X takes values in $\mathbb{R}$. The image measure $\mu_X = \mathbb{P} \circ X^{-1}$ is called the *law* or *distribution* of X. For real-valued random variables, μ_X is uniquely determined by its values on the π-system of intervals $(-\infty, x], x \in \mathbb{R}$, given by

$$F_X(x) = \mu_X((-\infty, x]) = \mathbb{P}(X \leq x).$$

The function F_X is called the *distribution function of* X.

Note that $F = F_X$ is increasing and right-continuous, with

$$\lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to \infty} F(x) = 1.$$

Let us call any function $F : \mathbb{R} \to [0, 1]$ satisfying these conditions a *distribution function*.

Set $\Omega = (0, 1]$ and $\mathcal{F} = \mathcal{B}((0, 1])$. Let $\mathbb{P}$ denote the restriction of Lebesgue measure to $\mathcal{F}$. Then $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Let F be any distribution function. Define $X : \Omega \to \mathbb{R}$ by

$$X(\omega) = \inf\{x : \omega \leq F(x)\}.$$

Then, by Lemma 2.2.1, X is a random variable and $X(\omega) \leq x$ if and only if $\omega \leq F(x)$. So

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}((0, F(x)]) = F(x).$$

Thus every distribution function is the distribution function of a random variable.

A countable family of random variables $(X_i : i \in I)$ is said to be *independent* if the σ-algebras $(\sigma(X_i) : i \in I)$ are independent. For a sequence $(X_n : n \in \mathbb{N})$ of real valued random variables, this is equivalent to the condition

$$\mathbb{P}(X_1 \leq x_1, \dots, X_n \leq x_n) = \mathbb{P}(X_1 \leq x_1) \dots \mathbb{P}(X_n \leq x_n)$$

for all $x_1, \dots, x_n \in \mathbb{R}$ and all n. A sequence of random variables $(X_n : n \geq 0)$ is often regarded as a *process* evolving in time. The σ-algebra generated by $X_0, \dots, X_n$

$$\mathcal{F}_n = \sigma(X_0, \dots, X_n)$$

contains those events depending (measurably) on $X_0, \dots, X_n$ and represents what is known about the process by time n.

2.4. **Rademacher functions.** We continue with the particular choice of probability space $(\Omega, \mathcal{F}, \mathbb{P})$ made in the preceding section. Provided that we forbid infinite sequences of 0's, each $\omega \in \Omega$ has a unique binary expansion

$$\omega = 0.\omega_1\omega_2\omega_3 \dots.$$

Define random variables $R_n : \Omega \to \{0, 1\}$ by $R_n(\omega) = \omega_n$. Then

$$R_1 = 1_{(\frac{1}{2},1]}, \quad R_2 = 1_{(\frac{1}{4},\frac{1}{2}]} + 1_{(\frac{3}{4},1]}, \quad R_3 = 1_{(\frac{1}{8},\frac{1}{4}]} + 1_{(\frac{3}{8},\frac{1}{2}]} + 1_{(\frac{5}{8},\frac{3}{4}]} + 1_{(\frac{7}{8},1]}.$$

These are called the *Rademacher functions*. The random variables $R_1, R_2, \dots$ are independent and *Bernoulli*, that is to say

$$\mathbb{P}(R_n = 0) = \mathbb{P}(R_n = 1) = 1/2.$$

The strong law of large numbers (proved in §10) applies here to show that

$$\mathbb{P}\left(\left\{\omega \in (0,1] : \frac{|\{k \le n : \omega_k = 1\}|}{n} \to \frac{1}{2}\right\}\right) = \mathbb{P}\left(\frac{R_1 + \dots + R_n}{n} \to \frac{1}{2}\right) = 1.$$

This is called Borel's normal number theorem: *almost every point in* $(0,1]$ *is* normal, *that is, has 'equal' proportions of* 0*'s and* 1*'s in its binary expansion.*

We now use a trick involving the Rademacher functions to construct on $\Omega = (0,1]$, not just one random variable, but an infinite sequence of independent random variables with given distribution functions.

Proposition 2.4.1. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be the probability space of Lebesgue measure on the Borel subsets of* $(0,1]$. *Let* $(F_n : n \in \mathbb{N})$ *be a sequence of distribution functions. Then there exists a sequence* $(X_n : n \in \mathbb{N})$ *of independent random variables on* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that* X_n *has distribution function* $F_{X_n} = F_n$ *for all* n.

Proof. Choose a bijection $m : \mathbb{N}^2 \to \mathbb{N}$ and set $Y_{k,n} = R_{m(k,n)}$, where R_m is the mth Rademacher function. Set

$$Y_n = \sum_{k=1}^{\infty} 2^{-k} Y_{k,n}.$$

Then $Y_1, Y_2, \dots$ are independent and, for all n, for $i2^{-k} = 0.y_1 \dots y_k$, we have

$$\mathbb{P}(i2^{-k} < Y_n \le (i+1)2^{-k}) = \mathbb{P}(Y_{1,n} = y_1, \dots, Y_{k,n} = y_k) = 2^{-k}$$

so $\mathbb{P}(Y_n \le x) = x$ for all $x \in (0,1]$. Set

$$G_n(y) = \inf\{x : y \le F_n(x)\}$$

then, by Lemma 2.2.1, G_n is Borel and $G_n(y) \le x$ if and only if $y \le F_n(x)$. So, if we set $X_n = G_n(Y_n)$, then $X_1, X_2, \dots$ are independent random variables on Ω and

$$\mathbb{P}(X_n \le x) = \mathbb{P}(G_n(Y_n) \le x) = \mathbb{P}(Y_n \le F_n(x)) = F_n(x).$$

□