2. Measurable functions and random variables

2.1. **Measurable functions.** Let $(E, \mathcal{E})$ and $(G, \mathcal{G})$ be measurable spaces. A function $f : E \to G$ is *measurable* if $f^{-1}(A) \in \mathcal{E}$ whenever $A \in \mathcal{G}$. Here $f^{-1}(A)$ denotes the *inverse image* of A by f

$$f^{-1}(A) = \{x \in E : f(x) \in A\}.$$

Usually $G = \mathbb{R}$ or $G = [-\infty, \infty]$, in which case $\mathcal{G}$ is always taken to be the Borel σ-algebra. If E is a topological space and $\mathcal{E} = \mathcal{B}(E)$, then a measurable function on E is called a *Borel* function. For any function $f : E \to G$, the inverse image preserves set operations

$$f^{-1}\left(\bigcup_i A_i\right) = \bigcup_i f^{-1}(A_i), \quad f^{-1}(G \setminus A) = E \setminus f^{-1}(A).$$

Therefore, the set $\{f^{-1}(A) : A \in \mathcal{G}\}$ is a σ-algebra on E and $\{A \subseteq G : f^{-1}(A) \in \mathcal{E}\}$ is a σ-algebra on G. In particular, if $\mathcal{G} = \sigma(\mathcal{A})$ and $f^{-1}(A) \in \mathcal{E}$ whenever $A \in \mathcal{A}$, then $\{A : f^{-1}(A) \in \mathcal{E}\}$ is a σ-algebra containing $\mathcal{A}$ and hence $\mathcal{G}$, so f is measurable. In the case $G = \mathbb{R}$, the Borel σ-algebra is generated by intervals of the form $(-\infty, y], y \in \mathbb{R}$, so, to show that $f : E \to \mathbb{R}$ is Borel measurable, it suffices to show that $\{x \in E : f(x) \leq y\} \in \mathcal{E}$ for all y.

If E is any topological space and $f : E \to \mathbb{R}$ is continuous, then $f^{-1}(U)$ is open in E and hence measurable, whenever U is open in $\mathbb{R}$; the open sets U generate $\mathcal{B}$, so *any continuous function is measurable.*

For $A \subseteq E$, the *indicator function* 1_A *of* A is the function $1_A : E \to \{0, 1\}$ which takes the value 1 on A and 0 otherwise. Note that *the indicator function of any measurable set is a measurable function.* Also, *the composition of measurable functions is measurable.*

Given any family of functions $f_i : E \to G, i \in I$, we can make them all measurable by taking

$$\mathcal{E} = \sigma(f_i^{-1}(A) : A \in \mathcal{G}, i \in I).$$

Then $\mathcal{E}$ is the *σ-algebra generated by* $(f_i : i \in I)$.

Proposition 2.1.1. *Let $f_n : E \to \mathbb{R}, n \in \mathbb{N}$, be measurable functions. Then so are $f_1 + f_2, f_1 f_2$ and each of the following:*

$$\inf_n f_n, \quad \sup_n f_n, \quad \liminf_n f_n, \quad \limsup_n f_n.$$

Theorem 2.1.2 (Monotone class theorem)**.** *Let $(E, \mathcal{E})$ be a measurable space and let $\mathcal{A}$ be a π-system generating $\mathcal{E}$. Let $\mathcal{V}$ be a vector space of bounded functions $f : E \to \mathbb{R}$ such that:*

- (i) *$1 \in \mathcal{V}$ and $1_A \in \mathcal{V}$ for all $A \in \mathcal{A}$;*
- (ii) *if $f_n \in \mathcal{V}$ for all n and f is bounded with $0 \leq f_n \uparrow f$, then $f \in \mathcal{V}$.*

Then $\mathcal{V}$ contains every bounded measurable function.

Proof. Consider $\mathcal{D} = \{A \in \mathcal{E} : 1_A \in \mathcal{V}\}$. Then $\mathcal{D}$ is a d-system containing $\mathcal{A}$, so $\mathcal{D} = \mathcal{E}$. Since $\mathcal{V}$ is a vector space, it thus contains all finite linear combinations of indicator functions of measurable sets. If f is a bounded and non-negative measurable function, then the functions $f_n = 2^{-n}\lfloor 2^n f \rfloor, n \in \mathbb{N}$, belong to $\mathcal{V}$ and $0 \le f_n \uparrow f$, so $f \in \mathcal{V}$. Finally, any bounded measurable function is the difference of two non-negative such functions, hence in $\mathcal{V}$. □

2.2. **Image measures.** Let $(E, \mathcal{E})$ and $(G, \mathcal{G})$ be measurable spaces and let μ be a measure on $\mathcal{E}$. Then any measurable function $f : E \to G$ induces an *image measure* $\nu = \mu \circ f^{-1}$ on $\mathcal{G}$, given by

$$\nu(A) = \mu(f^{-1}(A)).$$

We shall construct some new measures from Lebesgue measure in this way.

Lemma 2.2.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be non-constant, right-continuous and non-decreasing. Let $I = (g(-\infty), g(\infty))$ and define $f : I \to \mathbb{R}$ by $f(x) = \inf\{y \in \mathbb{R} : x \le g(y)\}$. Then f is left-continuous and non-decreasing. Moreover, for $x \in I$ and $y \in \mathbb{R}$,*

$$f(x) \le y \quad \textit{if and only if} \quad x \le g(y).$$

Proof. Fix $x \in I$ and consider the set $J_x = \{y \in \mathbb{R} : x \le g(y)\}$. Note that J_x is non-empty and is not the whole of $\mathbb{R}$. Since g is non-decreasing, if $y \in J_x$ and $y' \ge y$, then $y' \in J_x$. Since g is right-continuous, if $y_n \in J_x$ and $y_n \downarrow y$, then $y \in J_x$. Hence $J_x = [f(x), \infty)$ and $x \le g(y)$ if and only if $f(x) \le y$. For $x \le x'$, we have $J_x \supseteq J_{x'}$ and so $f(x) \le f(x')$. For $x_n \uparrow x$, we have $J_x = \cap_n J_{x_n}$, so $f(x_n) \to f(x)$. So f is left-continuous and non-decreasing, as claimed. □

Theorem 2.2.2. *Let $g : \mathbb{R} \to \mathbb{R}$ be non-constant, right-continuous and non-decreasing. Then there exists a unique Radon measure dg on $\mathbb{R}$ such that, for all $a, b \in \mathbb{R}$ with $a < b$,*

$$dg((a, b]) = g(b) - g(a).$$

Moreover, we obtain in this way all non-zero Radon measures on $\mathbb{R}$.

The measure dg is called the *Lebesgue-Stieltjes measure* associated with g.

Proof. Define I and f as in the lemma and let μ denote Lebesgue measure on I. Then f is Borel measurable and the induced measure $dg = \mu \circ f^{-1}$ on $\mathbb{R}$ satisfies

$$dg((a, b]) = \mu(\{x : f(x) > a \text{ and } f(x) \le b\}) = \mu((g(a), g(b)]) = g(b) - g(a).$$

The argument used for uniqueness of Lebesgue measure shows that there is at most one Borel measure with this property. Finally, if ν is any Radon measure on $\mathbb{R}$, we can define $g : \mathbb{R} \to \mathbb{R}$, right-continuous and non-decreasing, by

$$g(y) = \begin{cases} \nu((0, y]), & \text{if } y \ge 0, \\ -\nu((y, 0]), & \text{if } y < 0. \end{cases}$$

Then $\nu((a, b]) = g(b) - g(a)$ whenever $a < b$, so $\nu = dg$ by uniqueness. □