1.11. **Existence of a non-Lebesgue-measurable subset of $\mathbb{R}$.** For $x, y \in [0,1)$, let us write $x \sim y$ if $x - y \in \mathbb{Q}$. Then $\sim$ is an equivalence relation. Using the Axiom of Choice, we can find a subset S of $[0,1)$ containing exactly one representative of each equivalence class. Set $Q = \mathbb{Q} \cap [0,1)$ and, for each $q \in Q$, define $S_q = S + q = \{s + q \pmod 1 \colon s \in S\}$. It is an easy exercise to check that the sets S_q are all disjoint and their union is $[0,1)$.

Now, Lebesgue measure μ on $\mathcal{B} = \mathcal{B}([0,1))$ is translation invariant. That is to say, $\mu(B) = \mu(B + x)$ for all $B \in \mathcal{B}$ and all $x \in [0,1)$. If S were a Borel set, then we would have

$$1 = \mu([0,1)) = \sum_{q \in Q} \mu(S + q) = \sum_{q \in Q} \mu(S)$$

which is impossible. Hence $S \notin \mathcal{B}$.

A *Lebesgue measurable* set in $\mathbb{R}$ is any set of the form $A \cup N$, with A Borel and $N \subseteq B$ for some Borel set B with $\mu(B) = 0$. Thus the set of Lebesgue measurable sets is the *completion* of the Borel σ-algebra with respect to μ. See Exercise 1.9. The same argument shows that S cannot be Lebesgue measurable either.

1.12. **Independence.** A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ provides a model for an experiment whose outcome is subject to chance, according to the following interpretation:

- Ω is the set of possible outcomes
- $\mathcal{F}$ is the set of observable sets of outcomes, or *events*
- $\mathbb{P}(A)$ is the probability of the event A.

Relative to measure theory, probability theory is enriched by the significance attached to the notion of independence. Let I be a countable set. Say that events $A_i, i \in I$, are *independent* if, for all finite subsets $J \subseteq I$,

$$\mathbb{P}\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} \mathbb{P}(A_i).$$

Say that σ-algebras $\mathcal{A}_i \subseteq \mathcal{F}, i \in I$, are *independent* if $A_i, i \in I$, are independent whenever $A_i \in \mathcal{A}_i$ for all i. Here is a useful way to establish the independence of two σ-algebras.

Theorem 1.12.1. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be π-systems contained in $\mathcal{F}$ and suppose that*

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2)$$

whenever $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$. Then $\sigma(\mathcal{A}_1)$ and $\sigma(\mathcal{A}_2)$ are independent.

Proof. Fix $A_1 \in \mathcal{A}_1$ and define for $A \in \mathcal{F}$

$$\mu(A) = \mathbb{P}(A_1 \cap A), \quad \nu(A) = \mathbb{P}(A_1)\mathbb{P}(A).$$

Then μ and ν are measures which agree on the π-system $\mathcal{A}_2$, with $\mu(\Omega) = \nu(\Omega) = \mathbb{P}(A_1) < \infty$. So, by uniqueness of extension, for all $A_2 \in \sigma(\mathcal{A}_2)$,

$$\mathbb{P}(A_1 \cap A_2) = \mu(A_2) = \nu(A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2).$$

Now fix $A_2 \in \sigma(\mathcal{A}_2)$ and repeat the argument with

$$\mu'(A) = \mathbb{P}(A \cap A_2), \quad \nu'(A) = \mathbb{P}(A)\mathbb{P}(A_2)$$

to show that, for all $A_1 \in \sigma(\mathcal{A}_1)$,

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2).$$

□

1.13. **Borel-Cantelli lemmas.** Given a sequence of events $(A_n : n \in \mathbb{N})$, we may ask for the probability that infinitely many occur. Set

$$\limsup A_n = \bigcap_n \bigcup_{m \geq n} A_m, \quad \liminf A_n = \bigcup_n \bigcap_{m \geq n} A_m.$$

We sometimes write $\{A_n \text{ infinitely often}\}$ as an alternative for $\limsup A_n$, because $\omega \in \limsup A_n$ if and only if $\omega \in A_n$ for infinitely many n. Similarly, we write $\{A_n \text{ eventually}\}$ for $\liminf A_n$. The abbreviations i.o. and ev. are often used.

Lemma 1.13.1 (First Borel–Cantelli lemma). *If $\sum_n \mathbb{P}(A_n) < \infty$, then $\mathbb{P}(A_n \text{ i.o.}) = 0$.*

Proof. As $n \to \infty$ we have

$$\mathbb{P}(A_n \text{ i.o.}) \leq \mathbb{P}(\bigcup_{m \geq n} A_m) \leq \sum_{m \geq n} \mathbb{P}(A_m) \to 0.$$

□

We note that this argument is valid whether or not $\mathbb{P}$ is a probability measure.

Lemma 1.13.2 (Second Borel–Cantelli lemma). *Assume that the events $(A_n : n \in \mathbb{N})$ are independent. If $\sum_n \mathbb{P}(A_n) = \infty$, then $\mathbb{P}(A_n \text{ i.o.}) = 1$.*

Proof. We use the inequality $1 - a \leq e^{-a}$. Set $a_n = \mathbb{P}(A_n)$. Then, for all n we have

$$\mathbb{P}(\bigcap_{m \geq n} A_m^c) = \prod_{m \geq n} (1 - a_m) \leq \exp\{-\sum_{m \geq n} a_m\} = 0.$$

Hence $\mathbb{P}(A_n \text{ i.o.}) = 1 - \mathbb{P}(\bigcup_n \bigcap_{m \geq n} A_m^c) = 1$. □