1.7. **Uniqueness of measures.**

Theorem 1.7.1 (Uniqueness of extension)**.** *Let μ_1, μ_2 be measures on $(E, \mathcal{E})$ with $\mu_1(E) = \mu_2(E) < \infty$. Suppose that $\mu_1 = \mu_2$ on $\mathcal{A}$, for some π-system $\mathcal{A}$ generating $\mathcal{E}$. Then $\mu_1 = \mu_2$ on $\mathcal{E}$.*

Proof. Consider $\mathcal{D} = \{A \in \mathcal{E} : \mu_1(A) = \mu_2(A)\}$. By hypothesis, $E \in \mathcal{D}$; for $A, B \in \mathcal{E}$ with $A \subseteq B$, we have

$$\mu_1(A) + \mu_1(B \setminus A) = \mu_1(B) < \infty, \quad \mu_2(A) + \mu_2(B \setminus A) = \mu_2(B) < \infty$$

so, if $A, B \in \mathcal{D}$, then also $B \setminus A \in \mathcal{D}$; if $A_n \in \mathcal{D}, n \in \mathbb{N}$, with $A_n \uparrow A$, then

$$\mu_1(A) = \lim_n \mu_1(A_n) = \lim_n \mu_2(A_n) = \mu_2(A)$$

so $A \in \mathcal{D}$. Thus $\mathcal{D}$ is a d-system containing the π-system $\mathcal{A}$, so $\mathcal{D} = \mathcal{E}$ by Dynkin's lemma. □

1.8. **Borel sets and measures.** Let E be a topological space. The σ-algebra generated by the set of open sets is E is called the *Borel* σ-algebra of E and is denoted $\mathcal{B}(E)$. The Borel σ-algebra of $\mathbb{R}$ is denoted simply by $\mathcal{B}$. A measure μ on $(E, \mathcal{B}(E))$ is called a *Borel* measure on E. If moreover $\mu(K) < \infty$ for all compact sets K, then μ is called a *Radon* measure on E.

1.9. **Probability, finite and σ-finite measures.** If $\mu(E) = 1$ then μ is a *probability measure* and $(E, \mathcal{E}, \mu)$ is a *probability space*. The notation $(\Omega, \mathcal{F}, \mathbb{P})$ is often used to denote a probability space. If $\mu(E) < \infty$, then μ is a *finite* measure. If there exists a sequence of sets $(E_n : n \in \mathbb{N})$ in $\mathcal{E}$ with $\mu(E_n) < \infty$ for all n and $\bigcup_n E_n = E$, then μ is a *σ-finite* measure.

1.10. **Lebesgue measure.**

Theorem 1.10.1. *There exists a unique Borel measure μ on $\mathbb{R}$ such that, for all $a, b \in \mathbb{R}$ with $a < b$,*

$$\mu((a, b]) = b - a.$$

The measure μ is called *Lebesgue measure* on $\mathbb{R}$.

Proof. (*Existence.*) Consider the ring $\mathcal{A}$ of finite unions of disjoint intervals of the form

$$A = (a_1, b_1] \cup \cdots \cup (a_n, b_n].$$

We note that $\mathcal{A}$ generates $\mathcal{B}$. Define for such $A \in \mathcal{A}$

$$\mu(A) = \sum_{i=1}^{n} (b_i - a_i).$$

Note that the presentation of A is not unique, as $(a, b] \cup (b, c] = (a, c]$ whenever $a < b < c$. Nevertheless, it is easy to check that μ is well-defined and additive.

We aim to show that μ is countably additive on $\mathcal{A}$, which then proves existence by Carathéodory's extension theorem.

By additivity, it suffices to show that, if $A \in \mathcal{A}$ and if $(A_n : n \in \mathbb{N})$ is an increasing sequence in $\mathcal{A}$ with $A_n \uparrow A$, then $\mu(A_n) \to \mu(A)$. Set $B_n = A \setminus A_n$ then $B_n \in \mathcal{A}$ and $B_n \downarrow \emptyset$. By additivity again, it suffices to show that $\mu(B_n) \to 0$. Suppose, in fact, that for some $\varepsilon > 0$, we have $\mu(B_n) \geq 2\varepsilon$ for all n. For each n we can find $C_n \in \mathcal{A}$ with $\bar{C}_n \subseteq B_n$ and $\mu(B_n \setminus C_n) \leq \varepsilon 2^{-n}$. Then

$$\mu(B_n \setminus (C_1 \cap \cdots \cap C_n)) \leq \mu((B_1 \setminus C_1) \cup \cdots \cup (B_n \setminus C_n)) \leq \sum_{n \in \mathbb{N}} \varepsilon 2^{-n} = \varepsilon.$$

Since $\mu(B_n) \geq 2\varepsilon$, we must have $\mu(C_1 \cap \cdots \cap C_n) \geq \varepsilon$, so $C_1 \cap \cdots \cap C_n \neq \emptyset$, and so $K_n = \bar{C}_1 \cap \cdots \cap \bar{C}_n \neq \emptyset$. Now $(K_n : n \in \mathbb{N})$ is a decreasing sequence of bounded non-empty closed sets in $\mathbb{R}$, so $\emptyset \neq \bigcap_n K_n \subseteq \bigcap_n B_n$, which is a contradiction.

(*Uniqueness.*) Let λ be any measure on $\mathcal{B}$ with $\mu((a, b]) = b - a$ for all $a < b$. Fix n and consider

$$\mu_n(A) = \mu((n, n+1] \cap A), \quad \lambda_n(A) = \lambda((n, n+1] \cap A).$$

Then μ_n and λ_n are probability measures on $\mathcal{B}$ and $\mu_n = \lambda_n$ on the π-system of intervals of the form $(a, b]$, which generates $\mathcal{B}$. So, by Theorem 1.7.1, $\mu_n = \lambda_n$ on $\mathcal{B}$. Hence, for all $A \in \mathcal{B}$, we have

$$\mu(A) = \sum_n \mu_n(A) = \sum_n \lambda_n(A) = \lambda(A).$$

$\square$