10.3. **Central limit theorem.**

Theorem 10.3.1 (Central limit theorem)**.** *Let* $(X_n : n \in \mathbb{N})$ *be a sequence of independent, identically distributed, random variables with mean* 0 *and variance* 1*. Set* $S_n = X_1 + \cdots + X_n$*. Then, for all* $a < b$*, as* $n \to \infty$*,*

$$\mathbb{P}\left(\frac{S_n}{\sqrt{n}} \in [a,b]\right) \to \int_a^b \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.$$

Proof. Set $\phi(u) = \mathbb{E}(e^{iuX_1})$. Since $\mathbb{E}(X_1^2) < \infty$, we can differentiate $\mathbb{E}(e^{iuX_1})$ twice under the expectation, to show that

$$\phi(0) = 1, \quad \phi'(0) = 0, \quad \phi''(0) = -1.$$

Hence, by Taylor's theorem, as $u \to 0$,

$$\phi(u) = 1 - u^2/2 + o(u^2).$$

So, for the characteristic function ϕ_n of $S_n/\sqrt{n}$,

$$\phi_n(u) = \mathbb{E}(e^{iu(X_1+\cdots+X_n)/\sqrt{n}}) = \{\mathbb{E}(e^{i(u/\sqrt{n})X_1})\}^n = (1 - u^2/2n + o(u^2/n))^n.$$

The complex logarithm satisfies, as $z \to 0$,

$$\log(1+z) = z + o(|z|)$$

so, for each $u \in \mathbb{R}$, as $n \to \infty$,

$$\log \phi_n(u) = n \log(1 - u^2/2n + o(u^2/n)) = -u^2/2 + o(1).$$

Hence $\phi_n(u) \to e^{-u^2/2}$ for all u. But $e^{-u^2/2}$ is the characteristic function of the $N(0,1)$ distribution, so $S_n/\sqrt{n} \to N(0,1)$ in distribution by Theorem 7.5.1, as required. □

Here is an alternative argument, which does not rely on Lévy's continuity theorem. Take a random variable $Y \sim N(0,1)$, independent of the sequence $(X_n : n \in \mathbb{N})$. Fix $a < b$ and $\delta > 0$ and consider the function f which interpolates linearly the points $(-\infty, 0), (a-\delta, 0), (a, 1), (b, 1), (b+\delta, 0), (\infty, 0)$. Note that $|f(x+y) - f(x)| \leq |y|/\delta$ for all x, y. So, given $\varepsilon > 0$, for $t = (\pi/2)(\varepsilon\delta/3)^2$ and any random variable Z,

$$|\mathbb{E}(f(Z + \sqrt{t}Y)) - \mathbb{E}(f(Z))| \leq \mathbb{E}(\sqrt{t}|Y|)/\delta = \varepsilon/3.$$

Recall from the proof of the Fourier inversion formula that

$$\mathbb{E}\left(f\left(\frac{S_n}{\sqrt{n}} + \sqrt{t}Y\right)\right) = \int_{\mathbb{R}} \left(\frac{1}{2\pi}\int_{\mathbb{R}} \phi_n(u) e^{-u^2 t/2} e^{-iuy} du\right) f(y) dy.$$

Consider a second sequence of independent random variables $(\bar{X}_n : n \in \mathbb{N})$, also independent of Y, and with $\bar{X}_n \sim N(0,1)$ for all n. Note that $\bar{S}_n/\sqrt{n} \sim N(0,1)$ for all n. So

$$\mathbb{E}\left(f\left(\frac{\bar{S}_n}{\sqrt{n}} + \sqrt{t}Y\right)\right) = \int_{\mathbb{R}} \left(\frac{1}{2\pi}\int_{\mathbb{R}} e^{-u^2/2} e^{-u^2 t/2} e^{-iuy} du\right) f(y) dy.$$

Now $e^{-u^2t/2}f(y) \in L^1(du \otimes dy)$ and ϕ_n is bounded, with $\phi_n(u) \to e^{-u^2/2}$ for all u as $n \to \infty$, so, by dominated convergence, for n sufficiently large,

$$\left|\mathbb{E}\left(f\left(\frac{S_n}{\sqrt{n}} + \sqrt{t}Y\right)\right) - \mathbb{E}\left(f\left(\frac{\bar{S}_n}{\sqrt{n}} + \sqrt{t}Y\right)\right)\right| \le \varepsilon/3.$$

Hence, by taking $Z = S_n/\sqrt{n}$ and then $Z = \bar{S}_n/\sqrt{n}$, we obtain

$$\left|\mathbb{E}\left(f\left(S_n/\sqrt{n}\right)\right) - \mathbb{E}\left(f\left(\bar{S}_n/\sqrt{n}\right)\right)\right| \le \varepsilon.$$

But $\bar{S}_n/\sqrt{n} \sim Y$ for all n and $\varepsilon > 0$ is arbitrary, so we have shown that

$$\mathbb{E}(f(S_n/\sqrt{n})) \to \mathbb{E}(f(Y)) \quad \text{ as } n \to \infty.$$

The same argument applies to the function g, defined like f, but with a, b replaced by $a + \delta, b - \delta$ respectively. Now $g \le 1_{[a,b]} \le f$, so

$$\mathbb{E}\left(g\left(\frac{S_n}{\sqrt{n}}\right)\right) \le \mathbb{P}\left(\frac{S_n}{\sqrt{n}} \in [a,b]\right) \le \mathbb{E}\left(f\left(\frac{S_n}{\sqrt{n}}\right)\right).$$

On the other hand, as $\delta \downarrow 0$,

$$\mathbb{E}(g(Y)) \uparrow \int_a^b \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy, \quad \mathbb{E}(f(Y)) \downarrow \int_a^b \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

so we must have, as $n \to \infty$,

$$\mathbb{P}\left(\frac{S_n}{\sqrt{n}} \in [a,b]\right) \to \int_a^b \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.$$