Theorem 9.3.2 (Birkhoff's almost everywhere ergodic theorem)**.** *Assume that* $(E, \mathcal{E}, \mu)$ *is* σ*-finite and that* f *is an integrable function on* E*. Then there exists an invariant function* $\bar{f}$*, with* $\mu(|\bar{f}|) \leq \mu(|f|)$*, such that* $S_n(f)/n \to \bar{f}$ *a.e. as* $n \to \infty$*.*

Proof. The functions $\liminf_n(S_n/n)$ and $\limsup_n(S_n/n)$ are invariant. Therefore, for $a < b$, so is the following set

$$D = D(a,b) = \{\liminf_n(S_n/n) < a < b < \limsup_n(S_n/n)\}.$$

We shall show that $\mu(D) = 0$. First, by invariance, we can restrict everything to D and thereby reduce to the case $D = E$. Note that either $b > 0$ or $a < 0$. We can interchange the two cases by replacing f by $-f$. Let us assume then that $b > 0$.

Let $B \in \mathcal{E}$ with $\mu(B) < \infty$, then $g = f - b1_B$ is integrable and, for each $x \in D$, for some n,

$$S_n(g)(x) \geq S_n(f)(x) - nb > 0.$$

Hence $S^*(g) > 0$ everywhere and, by the maximal ergodic lemma,

$$0 \leq \int_D (f - b1_B)d\mu = \int_D f d\mu - b\mu(B).$$

Since μ is σ-finite, there is a sequence of sets $B_n \in \mathcal{E}$, with $\mu(B_n) < \infty$ for all n and $B_n \uparrow D$. Hence,

$$b\mu(D) = \lim_{n\to\infty} b\mu(B_n) \leq \int_D f d\mu.$$

In particular, we see that $\mu(D) < \infty$. A similar argument applied to $-f$ and $-a$, this time with $B = D$, shows that

$$(-a)\mu(D) \leq \int_D (-f) d\mu.$$

Hence

$$b\mu(D) \leq \int_D f d\mu \leq a\mu(D).$$

Since $a < b$ and the integral is finite, this forces $\mu(D) = 0$. Set

$$\Delta = \{\liminf_n(S_n/n) < \limsup_n(S_n/n)\}$$

then Δ is invariant. Also, $\Delta = \bigcup_{a,b\in\mathbb{Q},a<b} D(a,b)$, so $\mu(\Delta) = 0$. On the complement of Δ, S_n/n converges in $[-\infty, \infty]$, so we can define an invariant function $\bar{f}$ by

$$\bar{f} = \begin{cases} \lim_n(S_n/n) & \text{on } \Delta^c \\ 0 & \text{on } \Delta. \end{cases}$$

Finally, $\mu(|f \circ \theta^n|) = \mu(|f|)$, so $\mu(|S_n|) \leq n\mu(|f|)$ for all n. Hence, by Fatou's lemma,

$$\mu(|\bar{f}|) = \mu(\liminf_n |S_n/n|) \leq \liminf_n \mu(|S_n/n|) \leq \mu(|f|).$$

□

Theorem 9.3.3 (von Neumann's L^p ergodic theorem)**.** *Assume that $\mu(E) < \infty$. Let $p \in [1, \infty)$. Then, for all $f \in L^p(\mu)$, $S_n(f)/n \to \bar{f}$ in L^p.*

Proof. We have

$$\|f \circ \theta^n\|_p = \left(\int_E |f|^p \circ \theta^n d\mu \right)^{1/p} = \|f\|_p.$$

So, by Minkowski's inequality,

$$\|S_n(f)/n\|_p \leq \|f\|_p.$$

Given $\varepsilon > 0$, choose $K < \infty$ so that $\|f - g\|_p < \varepsilon/3$, where $g = (-K) \vee f \wedge K$. By Birkhoff's theorem, $S_n(g)/n \to \bar{g}$ a.e.. We have $|S_n(g)/n| \leq K$ for all n so, by bounded convergence, there exists N such that, for $n \geq N$,

$$\|S_n(g)/n - \bar{g}\|_p < \varepsilon/3.$$

By Fatou's lemma,

$$\begin{aligned} \|\bar{f} - \bar{g}\|_p^p &= \int_E \liminf_n |S_n(f - g)/n|^p d\mu \\ &\leq \liminf_n \int_E |S_n(f - g)/n|^p d\mu \leq \|f - g\|_p^p. \end{aligned}$$

Hence, for $n \geq N$,

$$\begin{aligned} \|S_n(f)/n - \bar{f}\|_p &\leq \|S_n(f - g)/n\|_p + \|S_n(g)/n - \bar{g}\|_p + \|\bar{g} - \bar{f}\|_p \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

□

Proof of Theorem 9.2.1. The details of showing that θ is measurable and measure-preserving are left as an exercise. To see that θ is ergodic, we recall the definition of the tail σ-algebras

$$\mathcal{T}_n = \sigma(X_m : m \geq n + 1), \quad \mathcal{T} = \bigcap_n \mathcal{T}_n.$$

For $A = \prod_{n \in \mathbb{N}} A_n \in \mathcal{A}$ we have

$$\theta^{-n}(A) = \{X_{n+k} \in A_k \text{ for all } k\} \in \mathcal{T}_n.$$

Since $\mathcal{T}_n$ is a σ-algebra, it follows that $\theta^{-n}(A) \in \mathcal{T}_n$ for all $A \in \mathcal{E}$, so $\mathcal{E}_\theta \subseteq \mathcal{T}$. Hence θ is ergodic by Kolmogorov's zero-one law. □