9. Ergodic theory

9.1. **Measure-preserving transformations.** Let $(E, \mathcal{E}, \mu)$ be a measure space. A measurable function $\theta : E \to E$ is called a *measure-preserving transformation* if

$$\mu(\theta^{-1}(A)) = \mu(A), \quad \text{for all } A \in \mathcal{E}.$$

A set $A \in \mathcal{E}$ is *invariant* if $\theta^{-1}(A) = A$. A measurable function f is *invariant* if $f = f \circ \theta$. The class of all invariant sets forms a σ-algebra, which we denote by $\mathcal{E}_\theta$. Then f is invariant if and only if f is $\mathcal{E}_\theta$-measurable. We say that θ is *ergodic* if $\mathcal{E}_\theta$ contains only sets of measure zero and their complements.

Here are two simple examples of measure preserving transformations.

(i) *Translation map on the torus.* Take $E = [0,1)^n$ with Lebesgue measure on its Borel σ-algebra, and consider addition modulo 1 in each coordinate. For $a \in E$ set

$$\theta_a(x_1, \ldots, x_n) = (x_1 + a_1, \ldots, x_n + a_n).$$

(ii) *Bakers' map.* Take $E = [0,1)$ with Lebesgue measure. Set

$$\theta(x) = 2x - \lfloor 2x \rfloor.$$

Proposition 9.1.1. *If f is integrable and θ is measure-preserving, then $f \circ \theta$ is integrable and*

$$\int_E f d\mu = \int_E f \circ \theta \, d\mu.$$

Proposition 9.1.2. *If θ is ergodic and f is invariant, then $f = c$ a.e., for some constant c.*

9.2. **Bernoulli shifts.** Let m be a probability measure on $\mathbb{R}$. In §2.4, we constructed a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which there exists a sequence of independent random variables $(Y_n : n \in \mathbb{N})$, all having distribution m. Consider now the infinite product space

$$E = \mathbb{R}^{\mathbb{N}} = \{x = (x_n : n \in \mathbb{N}) : x_n \in \mathbb{R} \text{ for all } n\}$$

and the σ-algebra $\mathcal{E}$ on E generated by the coordinate maps $X_n(x) = x_n$

$$\mathcal{E} = \sigma(X_n : n \in \mathbb{N}).$$

Note that $\mathcal{E}$ is also generated by the π-system

$$\mathcal{A} = \{\prod_{n \in \mathbb{N}} A_n : A_n \in \mathcal{B} \text{ for all } n,\ A_n = \mathbb{R} \text{ for sufficiently large } n\}.$$

Define $Y : \Omega \to E$ by $Y(\omega) = (Y_n(\omega) : n \in \mathbb{N})$. Then Y is measurable and the image measure $\mu = \mathbb{P} \circ Y^{-1}$ satisfies, for $A = \prod_{n \in \mathbb{N}} A_n \in \mathcal{A}$,

$$\mu(A) = \prod_{n \in \mathbb{N}} m(A_n).$$

By uniqueness of extension, μ is the unique measure on $\mathcal{E}$ having this property. Note that, under the probability measure μ, the coordinate maps $(X_n : n \in \mathbb{N})$ are themselves a sequence of independent random variables with law m. The probability space $(E, \mathcal{E}, \mu)$ is called the *canonical model* for such sequences. Define the *shift map* $\theta : E \to E$ by

$$\theta(x_1, x_2, \dots) = (x_2, x_3, \dots).$$

Theorem 9.2.1. *The shift map is an ergodic measure-preserving transformation.*

9.3. **Birkhoff's and von Neumann's ergodic theorems.** Throughout this section, $(E, \mathcal{E}, \mu)$ will denote a measure space, on which is given a measure-preserving transformation θ. Given an measurable function f, set $S_0 = 0$ and define, for $n \geq 1$,

$$S_n = S_n(f) = f + f \circ \theta + \cdots + f \circ \theta^{n-1}.$$

Lemma 9.3.1 (Maximal ergodic lemma)**.** *Let f be an integrable function on E. Set $S^* = \sup_{n \geq 0} S_n(f)$. Then*

$$\int_{\{S^* > 0\}} f d\mu \geq 0.$$

Proof. Set $S_n^* = \max_{0 \leq m \leq n} S_m$ and $A_n = \{S_n^* > 0\}$. Then, for $m = 1, \dots, n$,

$$S_m = f + S_{m-1} \circ \theta \leq f + S_n^* \circ \theta.$$

On A_n, we have $S_n^* = \max_{1 \leq m \leq n} S_m$, so

$$S_n^* \leq f + S_n^* \circ \theta.$$

On A_n^c, we have

$$S_n^* = 0 \leq S_n^* \circ \theta.$$

So, integrating and adding, we obtain

$$\int_E S_n^* d\mu \leq \int_{A_n} f d\mu + \int_E S_n^* \circ \theta d\mu.$$

But S_n^* is integrable, so

$$\int_E S_n^* \circ \theta d\mu = \int_E S_n^* d\mu < \infty$$

which forces

$$\int_{A_n} f d\mu \geq 0.$$

As $n \to \infty$, $A_n \uparrow \{S^* > 0\}$ so, by dominated convergence, with dominating function $|f|$,

$$\int_{\{S^* > 0\}} f d\mu = \lim_{n \to \infty} \int_{A_n} f d\mu \geq 0.$$

□