1.5. **Set functions and properties.** Let $\mathcal{A}$ be any set of subsets of E containing the empty set $\emptyset$. A *set function* is a function $\mu : \mathcal{A} \to [0, \infty]$ with $\mu(\emptyset) = 0$. Let μ be a set function. Say that μ is *increasing* if, for all $A, B \in \mathcal{A}$ with $A \subseteq B$,

$$\mu(A) \leq \mu(B).$$

Say that μ is *additive* if, for all disjoint sets $A, B \in \mathcal{A}$ with $A \cup B \in \mathcal{A}$,

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

Say that μ is *countably additive* if, for all sequences of disjoint sets $(A_n : n \in \mathbb{N})$ in $\mathcal{A}$ with $\bigcup_n A_n \in \mathcal{A}$,

$$\mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n).$$

Say that μ is *countably subadditive* if, for all sequences $(A_n : n \in \mathbb{N})$ in $\mathcal{A}$ with $\bigcup_n A_n \in \mathcal{A}$,

$$\mu\left(\bigcup_n A_n\right) \leq \sum_n \mu(A_n).$$

1.6. **Construction of measures.** Let $\mathcal{A}$ be a set of subsets of E. Say that $\mathcal{A}$ is a *ring* on E if $\emptyset \in \mathcal{A}$ and, for all $A, B \in \mathcal{A}$,

$$B \setminus A \in \mathcal{A}, \quad A \cup B \in \mathcal{A}.$$

Say that $\mathcal{A}$ is an *algebra* on E if $\emptyset \in \mathcal{A}$ and, for all $A, B \in \mathcal{A}$,

$$A^c \in \mathcal{A}, \quad A \cup B \in \mathcal{A}.$$

Theorem 1.6.1 (Carathéodory's extension theorem)**.** *Let $\mathcal{A}$ be a ring of subsets of E and let $\mu : \mathcal{A} \to [0, \infty]$ be a countably additive set function. Then μ extends to a measure on the σ-algebra generated by $\mathcal{A}$.*

Proof. For any $B \subseteq E$, define the *outer measure*

$$\mu^*(B) = \inf \sum_n \mu(A_n)$$

where the infimum is taken over all sequences $(A_n : n \in \mathbb{N})$ in $\mathcal{A}$ such that $B \subseteq \bigcup_n A_n$ and is taken to be ∞ if there is no such sequence. Note that μ^* is increasing and $\mu^*(\emptyset) = 0$. Let us say that $A \subseteq E$ is *μ^*-measurable* if, for all $B \subseteq E$,

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

Write $\mathcal{M}$ for the set of all μ^*-measurable sets. *We shall show that $\mathcal{M}$ is a σ-algebra containing $\mathcal{A}$ and that μ^* is a measure on $\mathcal{M}$, extending μ.* This will prove the theorem.

Step I. We show that μ^* is countably subadditive. Suppose that $B \subseteq \bigcup_n B_n$. If $\mu^*(B_n) < \infty$ for all n, then, given $\varepsilon > 0$, there exist sequences $(A_{nm} : m \in \mathbb{N})$ in $\mathcal{A}$, with

$$B_n \subseteq \bigcup_m A_{nm}, \quad \mu^*(B_n) + \varepsilon/2^n \geq \sum_m \mu(A_{nm}).$$

Then

$$B \subseteq \bigcup_n \bigcup_m A_{nm}$$

so

$$\mu^*(B) \leq \sum_n \sum_m \mu(A_{nm}) \leq \sum_n \mu^*(B_n) + \varepsilon.$$

Hence, in any case,

$$\mu^*(B) \leq \sum_n \mu^*(B_n).$$

Step II. We show that μ^* extends μ. Since $\mathcal{A}$ is a ring and μ is countably additive, μ is countably subadditive and increasing. Hence, for $A \in \mathcal{A}$ and any sequence $(A_n : n \in \mathbb{N})$ in $\mathcal{A}$ with $A \subseteq \bigcup_n A_n$,

$$\mu(A) \leq \sum_n \mu(A \cap A_n) \leq \sum_n \mu(A_n).$$

On taking the infimum over all such sequences, we see that $\mu(A) \leq \mu^*(A)$. On the other hand, it is obvious that $\mu^*(A) \leq \mu(A)$ for $A \in \mathcal{A}$.

Step III. We show that $\mathcal{M}$ contains $\mathcal{A}$. Let $A \in \mathcal{A}$ and $B \subseteq E$. We have to show that

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

By subadditivity of μ^*, it is enough to show that

$$\mu^*(B) \geq \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

If $\mu^*(B) = \infty$, this is clearly true, so let us assume that $\mu^*(B) < \infty$. Then, given $\varepsilon > 0$, we can find a sequence $(A_n : n \in \mathbb{N})$ in $\mathcal{A}$ such that

$$B \subseteq \bigcup_n A_n, \quad \mu^*(B) + \varepsilon \geq \sum_n \mu(A_n).$$

Then

$$B \cap A \subseteq \bigcup_n (A_n \cap A), \quad B \cap A^c \subseteq \bigcup_n (A_n \cap A^c)$$

so

$$\mu^*(B \cap A) + \mu^*(B \cap A^c) \leq \sum_n \mu(A_n \cap A) + \sum_n \mu(A_n \cap A^c) = \sum_n \mu(A_n) \leq \mu^*(B) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we are done.

Step IV. We show that $\mathcal{M}$ is an algebra. Clearly $E \in \mathcal{M}$ and $A^c \in \mathcal{M}$ whenever $A \in \mathcal{M}$. Suppose that $A_1, A_2 \in \mathcal{M}$ and $B \subseteq E$. Then

$$\begin{aligned}\mu^*(B) &= \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c) \\ &= \mu^*(B \cap A_1 \cap A_2) + \mu^*(B \cap A_1 \cap A_2^c) + \mu^*(B \cap A_1^c) \\ &= \mu^*(B \cap A_1 \cap A_2) + \mu^*(B \cap (A_1 \cap A_2)^c \cap A_1) + \mu^*(B \cap (A_1 \cap A_2)^c \cap A_1^c) \\ &= \mu^*(B \cap (A_1 \cap A_2)) + \mu^*(B \cap (A_1 \cap A_2)^c).\end{aligned}$$

Hence $A_1 \cap A_2 \in \mathcal{M}$.

Step V. We show that $\mathcal{M}$ is a σ-algebra and that μ^* is a measure on $\mathcal{M}$. We already know that $\mathcal{M}$ is an algebra, so it suffices to show that, for any sequence of disjoint sets $(A_n : n \in \mathbb{N})$ in $\mathcal{M}$, for $A = \bigcup_n A_n$ we have

$$A \in \mathcal{M}, \quad \mu^*(A) = \sum_n \mu^*(A_n).$$

So, take any $B \subseteq E$, then

$$\begin{aligned}\mu^*(B) &= \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c) \\ &= \mu^*(B \cap A_1) + \mu^*(B \cap A_2) + \mu^*(B \cap A_1^c \cap A_2^c) \\ &= \cdots = \sum_{i=1}^{n} \mu^*(B \cap A_i) + \mu^*(B \cap A_1^c \cap \cdots \cap A_n^c).\end{aligned}$$

Note that $\mu^*(B \cap A_1^c \cap \cdots \cap A_n^c) \geq \mu^*(B \cap A^c)$ for all n. Hence, on letting $n \to \infty$ and using countable subadditivity, we get

$$\mu^*(B) \geq \sum_{n=1}^{\infty} \mu^*(B \cap A_n) + \mu^*(B \cap A^c) \geq \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

The reverse inequality holds by subadditivity, so we have equality. Hence $A \in \mathcal{M}$ and, setting $B = A$, we get

$$\mu^*(A) = \sum_{n=1}^{\infty} \mu^*(A_n).$$

□