7.4. **Weak convergence.** Let F be a distribution function on $\mathbb{R}$ and let $(F_n : n \in \mathbb{N})$ be a sequence of such distribution functions. We say that $F_n \to F$ *as distribution functions* if $F_{X_n}(x) \to F_X(x)$ at every point $x \in \mathbb{R}$ where F_X is continuous.

Let μ be a Borel probability measure on $\mathbb{R}$ and let $(\mu_n : n \in \mathbb{N})$ be a sequence of such measures. We say that $\mu_n \to \mu$ *weakly* if $\mu_n(f) \to \mu(f)$ for all continuous bounded functions f on $\mathbb{R}$.

Let X be a random variable and let $(X_n : n \in \mathbb{N})$ be a sequence of random variables. We say that $X_n \to X$ *in distribution* if $F_{X_n} \to F_X$.

7.5. **Equivalent modes of convergence.** Suppose we are given a sequence of Borel probability measures $(\mu_n : n \in \mathbb{N})$ on $\mathbb{R}$ and a further such measure μ. Write F_n and F for the corresponding distribution functions, and write ϕ_n and ϕ for the corresponding Fourier transforms. On the probability space $([0,1), \mathcal{B}([0,1)), dx)$, define random variables $X_n(\omega) = \inf\{x \in \mathbb{R} : \omega \leq F_n(x)\}$ and $X(\omega) = \inf\{x \in \mathbb{R} : \omega \leq F(x)\}$, as in Subsection 2.3. Recall that $X_n \sim \mu_n$ and $X \sim \mu$.

Theorem 7.5.1. *The following are equivalent:*

(i) $\mu_n \to \mu$ *weakly,*
(ii) $F_n \to F$ *as distribution functions,*
(iii) $\phi_{X_n}(u) \to \phi_X(u)$ *for all* $u \in \mathbb{R}$,
(iv) $X_n \to X$ *a.s.*.

We do not prove this result in full, but note how to see certain of the implications. First, (iii) is a special case of (i), and (iv) implies (i) by bounded convergence, using $\mu_n(f) = \mathbb{E}(f(X_n)) \to \mathbb{E}(f(X)) = \mu(f)$. A proof that (ii) implies (iv) follows, with some care, from the definition of X_n and X. The fact that (iii) implies (ii) is a consequence of the following famous result, which we do not prove here.

Theorem 7.5.2 (Lévy's continuity theorem for characteristic functions)**.** *Let* $(X_n : n \in \mathbb{N})$ *be a sequence of random variables and suppose that* $\phi_{X_n}(u)$ *converges as* $n \to \infty$*, with limit* $\phi(u)$ *say, for all* $u \in \mathbb{R}$*. If* ϕ *is continuous in a neighbourhood of* 0*, then it is the characteristic function of some random variable* X*, and* $X_n \to X$ *in distribution.*

8. Gaussian random variables

8.1. **Gaussian random variables in** $\mathbb{R}$**.** A random variable X in $\mathbb{R}$ is *Gaussian* if, for some $\mu \in \mathbb{R}$ and some $\sigma^2 \in (0, \infty)$, X has density function

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}.$$

We also admit as Gaussian any random variable X with $X = \mu$ a.s., this degenerate case corresponding to taking $\sigma^2 = 0$. We write $X \sim N(\mu, \sigma^2)$.

Proposition 8.1.1. *Suppose* $X \sim N(\mu, \sigma^2)$ *and* $a, b \in \mathbb{R}$*. Then* (a) $\mathbb{E}(X) = \mu$, (b) $\mathrm{var}(X) = \sigma^2$, (c) $aX + b \sim N(a\mu + b, a^2\sigma^2)$, (d) $\phi_X(u) = e^{iu\mu - u^2\sigma^2/2}$.

8.2. **Gaussian random variables in $\mathbb{R}^n$.** A random variable X in $\mathbb{R}^n$ is *Gaussian* if $\langle u, X\rangle$ is Gaussian, for all $u \in \mathbb{R}^n$. An example of such a random variable is provided by $X = (X_1, \ldots, X_n)$, where $X_1, \ldots, X_n$ are independent $N(0,1)$ random variables. To see this, we note that

$$\mathbb{E}\, e^{i\langle u, X\rangle} = \mathbb{E} \prod_k e^{iu_k X_k} = e^{-|u|^2/2}$$

so $\langle u, X\rangle$ is $N(0, |u|^2)$ for all $u \in \mathbb{R}^n$.

Theorem 8.2.1. *Let X be a Gaussian random variable in $\mathbb{R}^n$. Then*

(a) *$AX + b$ is Gaussian, for all $n \times n$ matrices A and all $b \in \mathbb{R}^n$,*
(b) *$X \in L^2$ and its distribution is determined by its mean μ and its covariance matrix V,*
(c) *$\phi_X(u) = e^{i\langle u,\mu\rangle - \langle u, Vu\rangle/2}$,*
(d) *if V is invertible, then X has a density function on $\mathbb{R}^n$, given by*

$$f_X(x) = \frac{1}{\sqrt{\det(2\pi V)}} \exp\{-\langle x-\mu, V^{-1}(x-\mu)\rangle/2\},$$

(e) *if $X = (Y, Z)$, with Y in $\mathbb{R}^m$ and Z in $\mathbb{R}^p$, then*

$$\operatorname{cov}(Y, Z) = 0 \quad \textit{implies} \quad Y, Z \textit{ independent.}$$

Proof. For $u \in \mathbb{R}^n$, we have $\langle u, AX+b\rangle = \langle A^T u, X\rangle + \langle u, b\rangle$ so $\langle u, AX+b\rangle$ is Gaussian, by Proposition 8.1.1. This proves (a).

Each component X_k is Gaussian, so $X \in L^2$. Set $\mu = \mathbb{E}(X)$ and $V = \operatorname{var}(X)$. For $u \in \mathbb{R}^n$ we have $\mathbb{E}(\langle u, X\rangle) = \langle u, \mu\rangle$ and $\operatorname{var}(\langle u, X\rangle) = \operatorname{cov}(\langle u, X\rangle, \langle u, X\rangle) = \langle u, Vu\rangle$. Since $\langle u, X\rangle$ is Gaussian, by Proposition 8.1.1, we must have $\langle u, X\rangle \sim N(\langle u, \mu\rangle, \langle u, Vu\rangle)$ and $\phi_X(u) = \mathbb{E}\, e^{i\langle u, X\rangle} = e^{i\langle u,\mu\rangle - \langle u, Vu\rangle/2}$. This is (c) and (b) follows by uniqueness of characteristic functions.

Let $Y_1, \ldots, Y_n$ be independent $N(0,1)$ random variables. Then $Y = (Y_1, \ldots, Y_n)$ has density

$$f_Y(y) = (2\pi)^{-n/2} \exp\{-|y|^2/2\}.$$

Set $\tilde{X} = V^{1/2} Y + \mu$, then $\tilde{X}$ is Gaussian, with $\mathbb{E}(\tilde{X}) = \mu$ and $\operatorname{var}(\tilde{X}) = V$, so $\tilde{X} \sim X$. If V is invertible, then $\tilde{X}$ and hence X has the density claimed in (d), by a linear change of variables in $\mathbb{R}^n$.

Finally, if $X = (Y, Z)$ with $\operatorname{cov}(Y, Z) = 0$, then, for $u \in \mathbb{R}^m$ and $w \in \mathbb{R}^p$,

$$\langle (u, w), V(u, w)\rangle = \langle u, V_Y u\rangle + \langle w, V_Z w\rangle,$$

where $V_Y = \operatorname{var}(Y)$ and $V_Z = \operatorname{var}(Z)$. Then Y and Z are independent by Theorem 7.3.1, because the joint characteristic function $\phi_{Y,Z}$ splits as a product

$$\phi_{Y,Z}(u, w) = e^{i\langle u,\mu_Y\rangle - \langle u, V_Y u\rangle/2} e^{i\langle w,\mu_Z\rangle - \langle w, V_Z w\rangle/2} = \phi_Y(u)\phi_Z(w).$$

□