Theorem 7.2.2. *Let X be a random variable in $\mathbb{R}^n$. The law μ_X of X is uniquely determined by its characteristic function ϕ_X. Moreover, if ϕ_X is integrable, and we define*

$$f_X(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \phi_X(u) e^{-i\langle u,x\rangle} du,$$

then f_X is a continuous, bounded and non-negative function, which is a density function for X.

Proof. Let Z be a standard Gaussian random variable in $\mathbb{R}^n$, independent of X, and let g be a continuous function on $\mathbb{R}^n$ of compact support. Then, for any $t \in (0, \infty)$, by Fubini's theorem,

$$\mathbb{E}(g(X + \sqrt{t}Z)) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(x + \sqrt{t}z)(2\pi)^{-n/2} e^{-|z|^2/2} dz \mu_X(dx).$$

By the lemma, we have

$$\int_{\mathbb{R}^n} g(x + \sqrt{t}z)(2\pi)^{-n/2} e^{-|z|^2/2} dz = \mathbb{E}(g(x + \sqrt{t}Z))$$
$$= \int_{\mathbb{R}^n} g(y) p(t, x, y) dy = \int_{\mathbb{R}^n} g(y) \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle u,x\rangle} e^{-|u|^2 t/2} e^{-i\langle u,y\rangle} du dy,$$

so, by Fubini again,

$$\mathbb{E}(g(X + \sqrt{t}Z)) = \int_{\mathbb{R}^n} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \phi_X(u) e^{-|u|^2 t/2} e^{-i\langle u,y\rangle} du \right) g(y) dy.$$

By this formula, ϕ_X determines $\mathbb{E}(g(X+\sqrt{t}Z))$. For any such function g, by bounded convergence, we have

$$\mathbb{E}(g(X + \sqrt{t}Z)) \to \mathbb{E}(g(X))$$

as $t \downarrow 0$, so ϕ_X determines $\mathbb{E}(g(X))$. Hence ϕ_X determines μ_X.

Suppose now that ϕ_X is integrable. Then

$$|\phi_X(u)||g(y)| \in L^1(du \otimes dy).$$

So, by Fubini's theorem, $g.f_X \in L^1$ and, by dominated convergence, as $t \downarrow 0$,

$$\int_{\mathbb{R}^n} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \phi_X(u) e^{-|u|^2 t/2} e^{-i\langle u,y\rangle} du \right) g(y) dy$$
$$\to \int_{\mathbb{R}^n} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \phi_X(u) e^{-i\langle u,y\rangle} du \right) g(y) dy = \int_{\mathbb{R}^n} g(x) f_X(x) dx.$$

Hence we obtain the identity

$$\mathbb{E}(g(X)) = \int_{\mathbb{R}^n} g(x) f_X(x) dx.$$

Since $\overline{\phi_X(u)} = \phi_X(-u)$, we have $\bar{f}_X = f_X$, so f_X is real-valued. Moreover, f_X is continuous, by bounded convergence, and $\|f_X\|_\infty \leq (2\pi)^n \|\phi_X\|_1$. Since f_X is continuous, if it took a negative value anywhere, it would do so on an open interval of positive length, I say. There would exist a continuous function g, positive on I and vanishing outside I. Then we would have $\mathbb{E}(g(X)) \geq 0$ and $\int_{\mathbb{R}^n} g(x) f_X(x) dx < 0$, a contradiction. Hence, f_X is non-negative. It is now straightforward to extend the identity to all bounded Borel functions g, by a monotone class argument. In particular f_X is a density function for X. □

7.3. **Characteristic functions and independence.**

Theorem 7.3.1. *Let* $X = (X_1, \dots, X_n)$ *be a random variable in* $\mathbb{R}^n$. *Then the following are equivalent:*

(a) $X_1, \dots, X_n$ *are independent,*
(b) $\mu_X = \mu_{X_1} \otimes \dots \otimes \mu_{X_n}$,
(c) $\mathbb{E}\left(\prod_k f_k(X_k)\right) = \prod_k \mathbb{E}(f_k(X_k))$, *for all bounded Borel functions* $f_1, \dots, f_n$,
(d) $\phi_X(u) = \prod_k \phi_{X_k}(u_k)$, *for all* $u = (u_1, \dots, u_n) \in \mathbb{R}^n$.

Proof. If (a) holds, then

$$\mu_X(A_1 \times \dots \times A_n) = \prod_k \mu_{X_k}(A_k)$$

for all Borel sets $A_1, \dots, A_n$, so (b) holds, since this formula characterizes the product measure.

If (b) holds, then, for $f_1, \dots, f_n$ bounded Borel, by Fubini's theorem,

$$\mathbb{E}\left(\prod_k f_k(X_k)\right) = \int_{\mathbb{R}^n} \prod_k f_k(x_k) \mu_X(dx) = \prod_k \int_{\mathbb{R}} f_k(x_k) \mu_{X_k}(dx_k) = \prod_k \mathbb{E}(f_k(X_k)),$$

so (c) holds. Statement (d) is a special case of (c). Suppose, finally, that (d) holds and take independent random variables $\tilde{X}_1, \dots, \tilde{X}_n$ with $\mu_{\tilde{X}_k} = \mu_{X_k}$ for all k. We know that (a) implies (d), so

$$\phi_{\tilde{X}}(u) = \prod_k \phi_{\tilde{X}_k}(u_k) = \prod_k \phi_{X_k}(u_k) = \phi_X(u)$$

so $\mu_{\tilde{X}} = \mu_X$ by uniqueness of characteristic functions. Hence (a) holds. □