7. Characteristic functions and weak convergence

7.1. **Definitions.** For a finite Borel measure μ on $\mathbb{R}^n$, we define the *Fourier transform*

$$\hat{\mu}(u) = \int_{\mathbb{R}^n} e^{i\langle u,x\rangle}\mu(dx), \quad u \in \mathbb{R}^n.$$

Here, $\langle .,. \rangle$ denotes the usual inner product on $\mathbb{R}^n$. Note that $\hat{\mu}$ is in general complex-valued, with $\overline{\hat{\mu}(u)} = \hat{\mu}(-u)$, and $\|\hat{\mu}\|_\infty = \hat{\mu}(0) = \mu(\mathbb{R}^n)$. Moreover, by bounded convergence, $\hat{\mu}$ is continuous on $\mathbb{R}^n$.

For a random variable X in $\mathbb{R}^n$, we define the *characteristic function*

$$\phi_X(u) = \mathbb{E}(e^{i\langle u,X\rangle}), \quad u \in \mathbb{R}^n.$$

Thus $\phi_X = \hat{\mu}_X$, where μ_X is the law of X.

A random variable X in $\mathbb{R}^n$ is *standard Gaussian* if

$$\mathbb{P}(X \in A) = \int_A \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2} dx, \quad A \in \mathcal{B}.$$

Let us compute the characteristic function of a standard Gaussian random variable X in $\mathbb{R}$. We have

$$\phi_X(u) = \int_{\mathbb{R}} e^{iux} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = e^{-u^2/2} I$$

where

$$I = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-(x-iu)^2/2} dx.$$

The integral I can be evaluated by considering the integral of the analytic function $e^{-z^2/2}$ around the rectangular contour with corners $R, R - iu, -R - iu, -R$: by Cauchy's theorem, the integral round the contour vanishes, as do, in the limit $R \to \infty$, the contributions from the vertical sides of the rectangle. We deduce that

$$I = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1.$$

Hence $\phi_X(u) = e^{-u^2/2}$.

7.2. **Uniqueness and inversion.** We now show that a finite Borel measure is determined uniquely by its Fourier transform and obtain, where possible, an *inversion* formula by which to compute the measure from its transform.

Define, for $t > 0$ and $x, y \in \mathbb{R}^n$, the *heat kernel*

$$p(t,x,y) = \frac{1}{(2\pi t)^{n/2}} e^{-|y-x|^2/2t} = \prod_{k=1}^n \frac{1}{\sqrt{2\pi t}} e^{-|y_k - x_k|^2/2t}.$$

(This is the fundamental solution for the heat equation $(\partial/\partial t - \Delta)p = 0$ in $\mathbb{R}^n$, but we shall not pursue this property here.)

Lemma 7.2.1. *Let Z be a standard Gaussian random variable in $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$ and $t \in (0, \infty)$. Then*

(a) *the random variable $x + \sqrt{t}Z$ has density function $p(t, x, .)$ on $\mathbb{R}^n$,*

(b) *for all $y \in \mathbb{R}^n$, we have*

$$p(t, x, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle u,x\rangle} e^{-|u|^2 t/2} e^{-i\langle u,y\rangle} du.$$

Proof. The component random variables $Y_k = x_k + \sqrt{t}Z_k$ are independent Gaussians with mean x_k and variance t (see Subsection 8.1). So Y_k has density

$$\frac{1}{\sqrt{2\pi t}} e^{-|y_k - x_k|^2/2t}$$

and we obtain the claimed density function for Y as the product of the marginal densities.

For $u \in \mathbb{R}$ and $t \in (0, \infty)$, we know that

$$\int_{\mathbb{R}} e^{iux} \frac{1}{\sqrt{2\pi t}} e^{-|x|^2/2t} dx = \mathbb{E}(e^{iu\sqrt{t}Z_1}) = e^{u^2 t/2}.$$

By relabelling the variables we obtain, for $x_k, y_k, u_k \in \mathbb{R}$ and $t \in (0, \infty)$,

$$\int_{\mathbb{R}} e^{iu_k(x_k - y_k)} \frac{\sqrt{t}}{\sqrt{2\pi}} e^{-t|u_k|^2/2} du_k = e^{(x-y)^2/2t},$$

so

$$\frac{1}{\sqrt{2\pi t}} e^{-|y_k - x_k|^2/2t} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iu_k x_k} e^{-u_k^2 t/2} e^{-i\langle u_k, y_k\rangle} du_k.$$

On taking the product over $k \in \{1, \ldots, n\}$, we obtain the claimed formula for $p(t, x, y)$. □