6.2. **Uniform integrability.**

Lemma 6.2.1. *Let X be an integrable random variable and set*

$$I_X(\delta) = \sup\{\mathbb{E}(|X|1_A) : A \in \mathcal{F}, \mathbb{P}(A) \leq \delta\}.$$

Then $I_X(\delta) \downarrow 0$ as $\delta \downarrow 0$.

Proof. Suppose not. Then, for some $\varepsilon > 0$, there exist $A_n \in \mathcal{F}$, with $\mathbb{P}(A_n) \leq 2^{-n}$ and $\mathbb{E}(|X|1_{A_n}) \geq \varepsilon$ for all n. By the first Borel–Cantelli lemma, $\mathbb{P}(A_n \text{ i.o.}) = 0$. But then, by dominated convergence,

$$\varepsilon \leq \mathbb{E}(|X|1_{\bigcup_{m \geq n} A_m}) \to \mathbb{E}(|X|1_{\{A_n \text{ i.o.}\}}) = 0$$

which is a contradiction. □

Let $\mathcal{X}$ be a family of random variables. For $1 \leq p \leq \infty$, we say that $\mathcal{X}$ is *bounded in L^p* if $\sup_{X \in \mathcal{X}} \|X\|_p < \infty$. Let us define

$$I_{\mathcal{X}}(\delta) = \sup\{\mathbb{E}(|X|1_A) : X \in \mathcal{X}, A \in \mathcal{F}, \mathbb{P}(A) \leq \delta\}.$$

Obviously, $\mathcal{X}$ is bounded in L^1 if and only if $I_{\mathcal{X}}(1) < \infty$. We say that $\mathcal{X}$ is *uniformly integrable* or *UI* if $\mathcal{X}$ is bounded in L^1 and

$$I_{\mathcal{X}}(\delta) \downarrow 0, \quad \text{as } \delta \downarrow 0.$$

Note that, by Hölder's inequality, for conjugate indices $p, q \in (1, \infty)$,

$$\mathbb{E}(|X|1_A) \leq \|X\|_p (\mathbb{P}(A))^{1/q}.$$

Hence, if $\mathcal{X}$ is bounded in L^p, for some $p \in (1, \infty)$, then $\mathcal{X}$ is UI. The sequence $X_n = n1_{(0,1/n)}$ is bounded in L^1 for Lebesgue measure on $(0, 1]$, but not uniformly integrable.

Lemma 6.2.1 shows that any single integrable random variable is uniformly integrable. This extends easily to any finite collection of integrable random variables. Moreover, for any integrable random variable Y, the set

$$\mathcal{X} = \{X : X \text{ a random variable}, |X| \leq Y\}$$

is uniformly integrable, because $\mathbb{E}(|X|1_A) \leq \mathbb{E}(Y1_A)$ for all A.

The following result gives an alternative characterization of uniform integrability.

Lemma 6.2.2. *Let $\mathcal{X}$ be a family of random variables. Then $\mathcal{X}$ is UI if and only if*

$$\sup\{\mathbb{E}(|X|1_{|X| \geq K}) : X \in \mathcal{X}\} \to 0, \quad \text{as } K \to \infty.$$

Proof. Suppose $\mathcal{X}$ is *UI*. Given $\varepsilon > 0$, choose $\delta > 0$ so that $I_{\mathcal{X}}(\delta) < \varepsilon$, then choose $K < \infty$ so that $I_{\mathcal{X}}(1) \leq K\delta$. Then, for $X \in \mathcal{X}$ and $A = \{|X| \geq K\}$, we have $\mathbb{P}(A) \leq \delta$ so $\mathbb{E}(|X|1_A) < \varepsilon$. Hence, as $K \to \infty$,

$$\sup\{\mathbb{E}(|X|1_{|X| \geq K}) : X \in \mathcal{X}\} \to 0.$$

On the other hand, if this condition holds, then, since

$$\mathbb{E}(|X|) \leq K + \mathbb{E}(|X|1_{|X|\geq K}),$$

we have $I_{\mathcal{X}}(1) < \infty$. Given $\varepsilon > 0$, choose $K < \infty$ so that $\mathbb{E}(|X|1_{|X|\geq K}) < \varepsilon/2$ for all $X \in \mathcal{X}$. Then choose $\delta > 0$ so that $K\delta < \varepsilon/2$. For all $X \in \mathcal{X}$ and $A \in \mathcal{F}$ with $\mathbb{P}(A) < \delta$, we have

$$\mathbb{E}(|X|1_A) \leq \mathbb{E}(|X|1_{|X|\geq K}) + K\mathbb{P}(A) < \varepsilon.$$

Hence $\mathcal{X}$ is UI. □

Here is the definitive result on L^1-convergence of random variables.

Theorem 6.2.3. *Let X be a random variable and let $(X_n : n \in \mathbb{N})$ be a sequence of random variables. The following are equivalent:*

(a) *$X_n \in L^1$ for all n, $X \in L^1$ and $X_n \to X$ in L^1,*

(b) *$\{X_n : n \in \mathbb{N}\}$ is UI and $X_n \to X$ in probability.*

Proof. Suppose (a) holds. By Chebyshev's inequality, for $\varepsilon > 0$,

$$\mathbb{P}(|X_n - X| > \varepsilon) \leq \varepsilon^{-1}\mathbb{E}(|X_n - X|) \to 0$$

so $X_n \to X$ in probability. Moreover, given $\varepsilon > 0$, there exists N such that $\mathbb{E}(|X_n - X|) < \varepsilon/2$ whenever $n \geq N$. Then we can find $\delta > 0$ so that $\mathbb{P}(A) \leq \delta$ implies

$$\mathbb{E}(|X|1_A) \leq \varepsilon/2, \quad \mathbb{E}(|X_n|1_A) \leq \varepsilon, \quad n = 1, \dots, N.$$

Then, for $n \geq N$ and $\mathbb{P}(A) \leq \delta$,

$$\mathbb{E}(|X_n|1_A) \leq \mathbb{E}(|X_n - X|) + \mathbb{E}(|X|1_A) \leq \varepsilon.$$

Hence $\{X_n : n \in \mathbb{N}\}$ is UI. We have shown that (a) implies (b).

Suppose, on the other hand, that (b) holds. Then there is a subsequence (n_k) such that $X_{n_k} \to X$ a.s.. So, by Fatou's lemma, $\mathbb{E}(|X|) \leq \liminf_k \mathbb{E}(|X_{n_k}|) < \infty$. Now, given $\varepsilon > 0$, there exists $K < \infty$ such that, for all n,

$$\mathbb{E}(|X_n|1_{|X_n|\geq K}) < \varepsilon/3, \quad \mathbb{E}(|X|1_{|X|\geq K}) < \varepsilon/3.$$

Consider the uniformly bounded sequence $X_n^K = (-K) \vee X_n \wedge K$ and set $X^K = (-K) \vee X \wedge K$. Then $X_n^K \to X^K$ in probability, so, by bounded convergence, there exists N such that, for all $n \geq N$,

$$\mathbb{E}|X_n^K - X^K| < \varepsilon/3.$$

But then, for all $n \geq N$,

$$\mathbb{E}|X_n - X| \leq \mathbb{E}(|X_n|1_{|X_n|\geq K}) + \mathbb{E}|X_n^K - X^K| + \mathbb{E}(|X|1_{|X|\geq K}) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have shown that (b) implies (a). □