5. Completeness of L^p and orthogonal projection

5.1. **$\mathcal{L}^p$ as a Banach space.** Let V be a vector space. A map $v \mapsto \|v\| : V \to [0, \infty)$ is a *norm* if

(i) $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in V$,
(ii) $\|\alpha v\| = |\alpha|\|v\|$ for all $v \in V$ and $\alpha \in \mathbb{R}$,
(iii) $\|v\| = 0$ implies $v = 0$.

We note that, for any norm, if $\|v_n - v\| \to 0$ then $\|v_n\| \to \|v\|$.

A symmetric bilinear map $(u, v) \mapsto \langle u, v\rangle : V \times V \to \mathbb{R}$ is an *inner product* if $\langle v, v\rangle \geq 0$, with equality only if $v = 0$. For any inner product, $\langle ., .\rangle$, the map $v \mapsto \sqrt{\langle v, v\rangle}$ is a norm, by the Cauchy–Schwarz inequality.

Minkowski's inequality shows that each L^p space is a vector space and that the L^p-norms satisfy condition (i) above. Condition (ii) also holds. Condition (iii) fails, because $\|f\|_p = 0$ does not imply that $f = 0$, only that $f = 0$ a.e.. However, it is possible to make the L^p-norms into true norms by quotienting out by the subspace of measurable functions vanishing a.e.. This quotient will be denoted $\mathcal{L}^p$. Note that, for $f \in L^2$, we have $\|f\|_2^2 = \langle f, f\rangle$, where $\langle ., .\rangle$ is the symmetric bilinear form on L^2 given by

$$\langle f, g\rangle = \int_E fg d\mu.$$

Thus $\mathcal{L}^2$ is an inner product space. The notion of convergence in L^p defined in §4.1 is the usual notion of convergence in a normed space.

A normed vector space V is *complete* if every Cauchy sequence in V converges, that is to say, given any sequence $(v_n : n \in \mathbb{N})$ in V such that $\|v_n - v_m\| \to 0$ as $n, m \to \infty$, there exists $v \in V$ such that $\|v_n - v\| \to 0$ as $n \to \infty$. A complete normed vector space is called a *Banach space*. A complete inner product space is called a *Hilbert space*. Such spaces have many useful properties, which makes the following result important.

Theorem 5.1.1 (Completeness of L^p)**.** *Let $p \in [1, \infty]$. Let $(f_n : n \in \mathbb{N})$ be a sequence in L^p such that*

$$\|f_n - f_m\|_p \to 0 \quad \textit{as } n, m \to \infty.$$

Then there exists $f \in L^p$ such that

$$\|f_n - f\|_p \to 0 \quad \textit{as } n \to \infty.$$

Proof. Some modifications of the following argument are necessary in the case $p = \infty$, which are left as an exercise. We assume from now on that $p < \infty$. Choose a subsequence (n_k) such that

$$S = \sum_{k=1}^{\infty} \|f_{n_{k+1}} - f_{n_k}\|_p < \infty.$$

By Minkowski's inequality, for any $K \in \mathbb{N}$,

$$\|\sum_{k=1}^{K}|f_{n_{k+1}} - f_{n_k}|\|_p \leq S.$$

By monotone convergence this bound holds also for $K = \infty$, so

$$\sum_{k=1}^{\infty}|f_{n_{k+1}} - f_{n_k}| < \infty \quad \text{a.e..}$$

Hence, by completeness of $\mathbb{R}$, f_{n_k} converges a.e.. We define a measurable function f by

$$f(x) = \begin{cases} \lim f_{n_k}(x) & \text{if the limit exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Given $\varepsilon > 0$, we can find N so that $n \geq N$ implies

$$\mu(|f_n - f_m|^p) \leq \varepsilon, \quad \text{for all } m \geq n,$$

in particular $\mu(|f_n - f_{n_k}|^p) \leq \varepsilon$ for all sufficiently large k. Hence, by Fatou's lemma, for $n \geq N$,

$$\mu(|f_n - f|^p) = \mu(\liminf_k |f_n - f_{n_k}|^p) \leq \liminf_k \mu(|f_n - f_{n_k}|^p) \leq \varepsilon.$$

Hence $f \in L^p$ and, since $\varepsilon > 0$ was arbitrary, $\|f_n - f\|_p \to 0$. □

Corollary 5.1.2. *We have*

(a) $\mathcal{L}^p$ *is a Banach space, for all* $1 \leq p \leq \infty$*,*
(b) $\mathcal{L}^2$ *is a Hilbert space.*