4.1. **L^p-norms.** Let $(E, \mathcal{E}, \mu)$ be a measure space. For $1 \leq p < \infty$, we denote by $L^p = L^p(E, \mathcal{E}, \mu)$ the set of measurable functions f with finite L^p-*norm*:

$$\|f\|_p = \left(\int_E |f|^p d\mu \right)^{1/p} < \infty.$$

We denote by $L^\infty = L^\infty(E, \mathcal{E}, \mu)$ the set of measurable functions f with finite L^∞-*norm*:

$$\|f\|_\infty = \inf\{\lambda : |f| \leq \lambda \text{ a.e.}\}.$$

Note that $\|f\|_p \leq \mu(E)^{1/p}\|f\|_\infty$ for all $1 \leq p < \infty$. For $1 \leq p \leq \infty$ and $f_n \in L^p$, we say that f_n *converges to* f *in* L^p if $\|f_n - f\|_p \to 0$.

4.2. **Chebyshev's inequality.** Let f be a non-negative measurable function and let $\lambda \geq 0$. We use the notation $\{f \geq \lambda\}$ for the set $\{x \in E : f(x) \geq \lambda\}$. Note that

$$\lambda 1_{\{f \geq \lambda\}} \leq f$$

so on integrating we obtain *Chebyshev's inequality*

$$\lambda \mu(f \geq \lambda) \leq \mu(f).$$

Now let g be any measurable function. We can deduce inequalities for g by choosing some non-negative measurable function ϕ and applying Chebyshev's inequality to $f = \phi \circ g$. For example, if $g \in L^p, p < \infty$ and $\lambda > 0$, then

$$\mu(|g| \geq \lambda) = \mu(|g|^p \geq \lambda^p) \leq \lambda^{-p}\mu(|g|^p) < \infty.$$

So we obtain the *tail estimate*

$$\mu(|g| \geq \lambda) = O(\lambda^{-p}), \quad \text{as } \lambda \to \infty.$$

4.3. **Jensen's inequality.** Let $I \subseteq \mathbb{R}$ be an interval. A function $c : I \to \mathbb{R}$ is *convex* if, for all $x, y \in I$ and $t \in [0, 1]$,

$$c(tx + (1-t)y) \leq tc(x) + (1-t)c(y).$$

Lemma 4.3.1. *Let $c : I \to \mathbb{R}$ be convex and let m be a point in the interior of I. Then there exist $a, b \in \mathbb{R}$ such $c(x) \geq ax + b$ for all x, with equality at $x = m$.*

Proof. By convexity, for $m, x, y \in I$ with $x < m < y$, we have

$$\frac{c(m) - c(x)}{m - x} \leq \frac{c(y) - c(m)}{y - m}.$$

So, fixing an interior point m, there exists $a \in \mathbb{R}$ such that, for all $x < m$ and all $y > m$

$$\frac{c(m) - c(x)}{m - x} \leq a \leq \frac{c(y) - c(m)}{y - m}.$$

Then $c(x) \geq a(x - m) + c(m)$, for all $x \in I$. □

Theorem 4.3.2 (Jensen's inequality). *Let X be an integrable random variable with values in I and let $c : I \to \mathbb{R}$ be convex. Then $\mathbb{E}(c(X))$ is well defined and*

$$\mathbb{E}(c(X)) \geq c(\mathbb{E}(X)).$$

Proof. The case where X is almost surely constant is easy. We exclude it. Then $m = \mathbb{E}(X)$ must lie in the interior of I. Choose $a, b \in \mathbb{R}$ as in the lemma. Then $c(X) \geq aX + b$. In particular $\mathbb{E}(c(X)^-) \leq |a|\mathbb{E}(|X|) + |b| < \infty$, so $\mathbb{E}(c(X))$ is well defined. Moreover

$$\mathbb{E}(c(X)) \geq a\mathbb{E}(X) + b = am + b = c(m) = c(\mathbb{E}(X)).$$

□

We deduce from Jensen's inequality *the monotonicity of L^p-norms with respect to a probability measure*. Let $1 \leq p < q < \infty$. Set $c(x) = |x|^{q/p}$, then c is convex. So, for any $X \in L^p(\mathbb{P})$,

$$\|X\|_p = (\mathbb{E}|X|^p)^{1/p} = (c(\mathbb{E}|X|^p))^{1/q} \leq (\mathbb{E}\, c(|X|^p))^{1/q} = (\mathbb{E}|X|^q)^{1/q} = \|X\|_q.$$

In particular, $L^p(\mathbb{P}) \supseteq L^q(\mathbb{P})$.

4.4. **Hölder's inequality and Minkowski's inequality.** For $p, q \in [1, \infty]$, we say that p and q are *conjugate indices* if

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Theorem 4.4.1 (Hölder's inequality). *Let $p, q \in (1, \infty)$ be conjugate indices. Then, for all measurable functions f and g, we have*

$$\mu(|fg|) \leq \|f\|_p \|g\|_q.$$

Proof. The cases where $\|f\|_p = 0$ or $\|f\|_p = \infty$ are obvious. We exclude them. Then, by multiplying f by an appropriate constant, we are reduced to the case where $\|f\|_p = 1$. So we can define a probability measure $\mathbb{P}$ on $\mathcal{E}$ by

$$\mathbb{P}(A) = \int_A |f|^p d\mu.$$

For measurable functions $X \geq 0$,

$$\mathbb{E}(X) = \mu(X|f|^p), \quad \mathbb{E}(X) \leq \mathbb{E}(X^q)^{1/q}.$$

Note that $q(p-1) = p$. Then

$$\mu(|fg|) = \mu\left(\frac{|g|}{|f|^{p-1}}|f|^p\right) = \mathbb{E}\left(\frac{|g|}{|f|^{p-1}}\right) \leq \mathbb{E}\left(\frac{|g|^q}{|f|^{q(p-1)}}\right)^{1/q} = \mu(|g|^q)^{1/q} = \|f\|_p\|g\|_q.$$

□

Theorem 4.4.2 (Minkowski's inequality)**.** *For* $p \in [1, \infty)$ *and measurable functions* f *and* g*, we have*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof. The cases where $p = 1$ or where $\|f\|_p = \infty$ or $\|g\|_p = \infty$ are easy. We exclude them. Then, since $|f + g|^p \leq 2^p(|f|^p + |g|^p)$, we have

$$\mu(|f + g|^p) \leq 2^p\{\mu(|f|^p) + \mu(|g|^p)\} < \infty.$$

The case where $\|f + g\|_p = 0$ is clear, so let us assume $\|f + g\|_p > 0$. Observe that

$$\||f + g|^{p-1}\|_q = \mu(|f + g|^{(p-1)q})^{1/q} = \mu(|f + g|^p)^{1-1/p}.$$

So, by Hölder's inequality,

$$\begin{aligned} \mu(|f + g|^p) &\leq \mu(|f||f + g|^{p-1}) + \mu(|g||f + g|^{p-1}) \\ &\leq (\|f\|_p + \|g\|_p)\||f + g|^{p-1}\|_q. \end{aligned}$$

The result follows on dividing both sides by $\||f + g|^{p-1}\|_q$. $\square$