3.6. **Product measure and Fubini's theorem.** Let $(E_1, \mathcal{E}_1, \mu_1)$ and $(E_2, \mathcal{E}_2, \mu_2)$ be *finite* measure spaces. The set

$$\mathcal{A} = \{A_1 \times A_2 : A_1 \in \mathcal{E}_1, A_2 \in \mathcal{E}_2\}$$

is a π-system of subsets of $E = E_1 \times E_2$. Define the *product σ-algebra*

$$\mathcal{E}_1 \otimes \mathcal{E}_2 = \sigma(\mathcal{A}).$$

Set $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$.

Lemma 3.6.1. *Let $f : E \to \mathbb{R}$ be $\mathcal{E}$-measurable. Then, for all $x_1 \in E_1$, the function $x_2 \mapsto f(x_1, x_2) : E_2 \to \mathbb{R}$ is $\mathcal{E}_2$-measurable.*

Proof. Denote by $\mathcal{V}$ the set of bounded $\mathcal{E}$-measurable functions for which the conclusion holds. Then $\mathcal{V}$ is a vector space, containing the indicator function 1_A of every set $A \in \mathcal{A}$. Moreover, if $f_n \in \mathcal{V}$ for all n and if f is bounded with $0 \leq f_n \uparrow f$, then also $f \in \mathcal{V}$. So, by the monotone class theorem, $\mathcal{V}$ contains all bounded $\mathcal{E}$-measurable functions. The rest is easy. □

Lemma 3.6.2. *For all bounded $\mathcal{E}$-measurable functions f, the function*

$$x_1 \mapsto f_1(x_1) = \int_{E_2} f(x_1, x_2)\mu_2(dx_2) : E_1 \to \mathbb{R}$$

is bounded and $\mathcal{E}_1$-measurable.

Proof. Apply the monotone class theorem, as in the preceding lemma. Note that finiteness of μ_1 and μ_2 is essential to the argument. □

Theorem 3.6.3 (Product measure)**.** *There exists a unique measure $\mu = \mu_1 \otimes \mu_2$ on $\mathcal{E}$ such that*

$$\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$$

for all $A_1 \in \mathcal{E}_1$ and $A_2 \in \mathcal{E}_2$.

Proof. Uniqueness holds because $\mathcal{A}$ is a π-system generating $\mathcal{E}$. For existence, by the lemmas, we can define

$$\mu(A) = \int_{E_1} \left(\int_{E_2} 1_A(x_1, x_2)\mu_2(dx_2) \right) \mu_1(dx_1)$$

and use monotone convergence to see that μ is countably additive. □

Proposition 3.6.4. *Let $\hat{\mathcal{E}} = \mathcal{E}_2 \otimes \mathcal{E}_1$ and $\hat{\mu} = \mu_2 \otimes \mu_1$. For a function f on $E_1 \times E_2$, write $\hat{f}$ for the function on $E_2 \times E_1$ given by $\hat{f}(x_2, x_1) = f(x_1, x_2)$. Suppose that f is $\mathcal{E}$-measurable. Then $\hat{f}$ is $\hat{\mathcal{E}}$-measurable, and if f is also non-negative, then $\hat{\mu}(\hat{f}) = \mu(f)$.*

Theorem 3.6.5 (Fubini's theorem).

(a) *Let f be $\mathcal{E}$-measurable and non-negative. Then*

$$\mu(f) = \int_{E_1} \left(\int_{E_2} f(x_1, x_2) \mu_2(dx_2) \right) \mu_1(dx_1).$$

(b) *Let f be μ-integrable. Then*

(i) *$x_2 \mapsto f(x_1, x_2)$ is μ_2-integrable for μ_1-almost all x_1,*

(ii) *$x_1 \mapsto \int_{E_2} f(x_1, x_2)\mu_2(dx_2)$ is μ_1-integrable*

and the formula for $\mu(f)$ in (a) *holds.*

Note that the *iterated integral* in (a) is well defined, for all bounded or non-negative measurable functions f, by Lemmas 3.6.1 and 3.6.2. Note also that, in combination wih Proposition 3.6.4, Fubini's theorem allows us to interchange the order of integration in multiple integrals,whenever the integrand is non-negative or μ-integrable.

Proof. Denote by $\mathcal{V}$ the set of all bounded $\mathcal{E}$-measurable functions f for which the formula holds. Then $\mathcal{V}$ contains the indicator function of every $\mathcal{E}$-measurable set so, by the monotone class theorem, $\mathcal{V}$ contains all bounded $\mathcal{E}$-measurable functions. Hence, for all $\mathcal{E}$-measurable functions f, we have

$$\mu(f_n) = \int_{E_1} \left(\int_{E_2} f_n(x_1, x_2) \mu_2(dx_2) \right) \mu_1(dx_1)$$

where $f_n = (-n) \vee f \wedge n$.

For f non-negative, we can pass to the limit as $n \to \infty$ by monotone convergence to extend the formula to f. That proves (a).

If f is μ-integrable, then, by (a)

$$\int_{E_1} \left(\int_{E_2} |f(x_1, x_2)| \mu_2(dx_2) \right) \mu_1(dx_1) = \mu(|f|) < \infty.$$

Hence we obtain (i) and (ii). Then, by dominated convergence, we can pass to the limit as $n \to \infty$ in the formula for $\mu(f_n)$ to obtain the desired formula for $\mu(f)$. □

The existence of product measure and Fubini's theorem extend easily to σ-finite measure spaces. The operation of taking the product of two measure spaces is associative, by a π-system uniqueness argument. So we can, by induction, take the product of a finite number, without specifying the order. The measure obtained by taking the n-fold product of Lebesgue measure on $\mathbb{R}$ is called *Lebesgue measure on $\mathbb{R}^n$*. The corresponding integral is written

$$\int_{\mathbb{R}^n} f(x)dx.$$