Proposition 3.4.2. *Let $\phi : [a, b] \to \mathbb{R}$ be continuously differentiable and strictly increasing. Then, for all non-negative Borel functions g on $[\phi(a), \phi(b)]$,*

$$\int_{\phi(a)}^{\phi(b)} g(y)dy = \int_a^b g(\phi(x))\phi'(x)dx.$$

The proposition can be proved as follows. First, the case where g is the indicator function of an interval follows from the Fundamental Theorem of Calculus. Next, show that the set of Borel sets B such that the conclusion holds for $g = 1_B$ is a d-system, which must then be the whole Borel σ-algebra by Dynkin's lemma. The identity extends to simple functions by linearity and then to all non-negative measurable functions g by monotone convergence, using approximating simple functions $(2^{-n}\lfloor 2^n g \rfloor) \wedge n$.

An general formulation of this procedure, which is often used, is given in the monotone class theorem Theorem 2.1.2.

3.5. **Differentiation under the integral sign.** Integration in one variable and differentiation in another can be interchanged subject to some regularity conditions.

Theorem 3.5.1 (Differentiation under the integral sign)**.** *Let $U \subseteq \mathbb{R}$ be open and suppose that $f : U \times E \to \mathbb{R}$ satisfies:*

(i) *$x \mapsto f(t,x)$ is integrable for all t,*
(ii) *$t \mapsto f(t,x)$ is differentiable for all x,*
(iii) *for some integrable function g, for all $x \in E$ and all $t \in U$,*

$$\left|\frac{\partial f}{\partial t}(t,x)\right| \leq g(x).$$

Then the function $x \mapsto (\partial f/\partial t)(t,x)$ is integrable for all t. Moreover, the function $F : U \to \mathbb{R}$, defined by

$$F(t) = \int_E f(t,x)\mu(dx),$$

is differentiable and

$$\frac{d}{dt}F(t) = \int_E \frac{\partial f}{\partial t}(t,x)\mu(dx).$$

Proof. Take any sequence $h_n \to 0$ and set

$$g_n(x) = \frac{f(t+h_n,x) - f(t,x)}{h_n} - \frac{\partial f}{\partial t}(t,x).$$

Then $g_n(x) \to 0$ for all $x \in E$ and, by the mean value theorem, $|g_n| \leq 2g$ for all n. In particular, for all t, the function $x \mapsto (\partial f/\partial t)(t,x)$ is the limit of measurable functions, hence measurable, and hence integrable, by (iii).Then, by dominated convergence,

$$\frac{F(t+h_n) - F(t)}{h_n} - \int_E \frac{\partial f}{\partial t}(t,x)\mu(dx) = \int_E g_n(x)\mu(dx) \to 0.$$

□