3.3. **Transformations of integrals.**

Proposition 3.3.1. *Let $(E, \mathcal{E}, \mu)$ be a measure space and let $A \in \mathcal{E}$. Then the set $\mathcal{E}_A$ of measurable subsets of A is a σ-algebra and the restriction μ_A of μ to $\mathcal{E}_A$ is a measure. Moreover, for any non-negative measurable function f on E, we have*

$$\mu(f1_A) = \mu_A(f|_A).$$

In the case of Lebesgue measure on $\mathbb{R}$, we write, for any interval I with $\inf I = a$ and $\sup I = b$,

$$\int_{\mathbb{R}} f1_I(x)dx = \int_I f(x)dx = \int_a^b f(x)dx.$$

Note that the sets $\{a\}$ and $\{b\}$ have measure zero, so we do not need to specify whether they are included in I or not.

Proposition 3.3.2. *Let $(E, \mathcal{E})$ and $(G, \mathcal{G})$ be measure spaces and let $f : E \to G$ be a measurable function. Given a measure μ on $(E, \mathcal{E})$, define $\nu = \mu \circ f^{-1}$, the image measure on $(G, \mathcal{G})$. Then, for all non-negative measurable functions g on G,*

$$\nu(g) = \mu(g \circ f).$$

In particular, for a G-valued random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, for any non-negative measurable function g on G, we have

$$\mathbb{E}(g(X)) = \mu_X(g).$$

Proposition 3.3.3. *Let $(E, \mathcal{E}, \mu)$ be a measure space and let f be a non-negative measurable function on E. Define $\nu(A) = \mu(f1_A), A \in \mathcal{E}$. Then ν is a measure on E and, for all non-negative measurable functions g on E,*

$$\nu(g) = \mu(fg).$$

In particular, to each non-negative Borel function f on $\mathbb{R}$, there corresponds a Borel measure μ on $\mathbb{R}$ given by $\mu(A) = \int_A f(x)dx$. Then, for all non-negative Borel functions g,

$$\mu(g) = \int_{\mathbb{R}^n} g(x)f(x)dx.$$

We say that μ *has density f (with respect to Lebesgue measure)*.

If the law μ_X of a real-valued random variable X has a density f_X, then we call f_X a *density function* for X. Then $\mathbb{P}(X \in A) = \int_A f_X(x)dx$, for all Borel sets A, and, for for all non-negative Borel functions g on $\mathbb{R}$,

$$\mathbb{E}(g(X)) = \mu_X(g) = \int_{\mathbb{R}} g(x)f_X(x)dx.$$

3.4. **Fundamental theorem of calculus.** We show that integration with respect to Lebesgue measure on $\mathbb{R}$ acts as an inverse to differentiation. Since we restrict here to the integration of continuous functions, the proof is the same as for the Riemann integral.

Theorem 3.4.1 (Fundamental theorem of calculus)**.**

(i) *Let $f : [a, b] \to \mathbb{R}$ be a continuous function and set*

$$F_a(t) = \int_a^t f(x)dx.$$

Then F_a is differentiable on $[a, b]$, with $F_a' = f$.

(ii) *Let $F : [a, b] \to \mathbb{R}$ be differentiable with continuous derivative f. Then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. Fix $t \in [a, b)$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(t)| \leq \varepsilon$ whenever $|x - t| \leq \delta$. So, for $0 < h \leq \delta$,

$$\left|\frac{F_a(t+h) - F_a(t)}{h} - f(t)\right| = \frac{1}{h}\left|\int_t^{t+h} (f(x) - f(t))dx\right|$$
$$\leq \frac{1}{h}\int_t^{t+h} |f(x) - f(t)|dx \leq \frac{\varepsilon}{h}\int_t^{t+h} dx = \varepsilon.$$

Hence F_a is differentiable on the right at t with derivative $f(t)$. Similarly, for all $t \in (a, b]$, F_a is differentiable on the left at t with derivative $f(t)$. Finally, $(F - F_a)'(t) = 0$ for all $t \in (a, b)$ so $F - F_a$ is constant (by the mean value theorem), and so

$$F(b) - F(a) = F_a(b) - F_a(a) = \int_a^b f(x)dx.$$

□

Proposition 3.4.2. *Let $\phi : [a, b] \to \mathbb{R}$ be continuously differentiable and strictly increasing. Then, for all non-negative Borel functions g on $[\phi(a), \phi(b)]$,*

$$\int_{\phi(a)}^{\phi(b)} g(y)dy = \int_a^b g(\phi(x))\phi'(x)dx.$$