1. Measures

1.1. **Definitions.** Let E be a set. A *σ-algebra* $\mathcal{E}$ on E is a set of subsets of E, containing the empty set $\emptyset$ and such that, for all $A \in \mathcal{E}$ and all sequences $(A_n : n \in \mathbb{N})$ in $\mathcal{E}$,

$$A^c \in \mathcal{E}, \quad \bigcup_n A_n \in \mathcal{E}.$$

The pair $(E, \mathcal{E})$ is called a *measurable space*. Given $(E, \mathcal{E})$, each $A \in \mathcal{E}$ is called a *measurable set*.

A *measure* μ on $(E, \mathcal{E})$ is a function $\mu : \mathcal{E} \to [0, \infty]$, with $\mu(\emptyset) = 0$, such that, for any sequence $(A_n : n \in \mathbb{N})$ of disjoint elements of $\mathcal{E}$,

$$\mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n).$$

The triple $(E, \mathcal{E}, \mu)$ is called a *measure space*.

1.2. **Discrete measure theory.** Let E be a countable set and let $\mathcal{E} = \mathcal{P}(E)$. A *mass function* is any function $m : E \to [0, \infty]$. If μ is a measure on $(E, \mathcal{E})$, then, by countable additivity,

$$\mu(A) = \sum_{x \in A} \mu(\{x\}), \quad A \subseteq E.$$

So there is a one-to-one correspondence between measures and mass functions, given by

$$m(x) = \mu(\{x\}), \quad \mu(A) = \sum_{x \in A} m(x).$$

This sort of measure space provides a 'toy' version of the general theory, where each of the results we prove for general measure spaces reduces to some straightforward fact about the convergence of series. This is all one needs to do elementary discrete probability and discrete-time Markov chains, so these topics are usually introduced without discussing measure theory.

Discrete measure theory is essentially the only context where one can define a measure explicitly, because, in general, σ-algebras are not amenable to an explicit presentation which would allow us to make such a definition. Instead one specifies the values to be taken on some smaller set of subsets, which generates the σ-algebra. This gives rise to two problems: first to know that there is a measure extending the given set function, second to know that there is not more than one. The first problem, which is one of construction, is often dealt with by Carathéodory's extension theorem. The second problem, that of uniqueness, is often dealt with by Dynkin's π-system lemma.

1.3. **Generated σ-algebras.** Let $\mathcal{A}$ be a set of subsets of E. Define

$$\sigma(\mathcal{A}) = \{A \subseteq E : A \in \mathcal{E} \quad \text{for all } \sigma\text{-algebras } \mathcal{E} \text{ containing } \mathcal{A}\}.$$

Then $\sigma(\mathcal{A})$ is a σ-algebra, which is called the *σ-algebra generated by $\mathcal{A}$* . It is the smallest σ-algebra containing $\mathcal{A}$.

1.4. **π-systems and d-systems.** Let $\mathcal{A}$ be a set of subsets of E. Say that $\mathcal{A}$ is a *π-system* if $\emptyset \in \mathcal{A}$ and, for all $A, B \in \mathcal{A}$,

$$A \cap B \in \mathcal{A}.$$

Say that $\mathcal{A}$ is a *d-system* if $E \in \mathcal{A}$ and, for all $A, B \in \mathcal{A}$ with $A \subseteq B$ and all increasing sequences $(A_n : n \in \mathbb{N})$ in $\mathcal{A}$,

$$B \setminus A \in \mathcal{A}, \quad \bigcup_n A_n \in \mathcal{A}.$$

Note that, if $\mathcal{A}$ is both a π-system and a d-system, then $\mathcal{A}$ is a σ-algebra.

Lemma 1.4.1 (Dynkin's π-system lemma)**.** *Let $\mathcal{A}$ be a π-system. Then any d-system containing $\mathcal{A}$ contains also the σ-algebra generated by $\mathcal{A}$.*

Proof. Denote by $\mathcal{D}$ the intersection of all d-systems containing $\mathcal{A}$. Then $\mathcal{D}$ is itself a d-system. We shall show that $\mathcal{D}$ is also a π-system and hence a σ-algebra, thus proving the lemma. Consider

$$\mathcal{D}' = \{B \in \mathcal{D} : B \cap A \in \mathcal{D} \text{ for all } A \in \mathcal{A}\}.$$

Then $\mathcal{A} \subseteq \mathcal{D}'$ because $\mathcal{A}$ is a π-system. Let us check that $\mathcal{D}'$ is a d-system: clearly $E \in \mathcal{D}'$; next, suppose $B_1, B_2 \in \mathcal{D}'$ with $B_1 \subseteq B_2$, then for $A \in \mathcal{A}$ we have

$$(B_2 \setminus B_1) \cap A = (B_2 \cap A) \setminus (B_1 \cap A) \in \mathcal{D}$$

because $\mathcal{D}$ is a d-system, so $B_2 \setminus B_1 \in \mathcal{D}'$; finally, if $B_n \in \mathcal{D}', n \in \mathbb{N}$, and $B_n \uparrow B$, then for $A \in \mathcal{A}$ we have

$$B_n \cap A \uparrow B \cap A$$

so $B \cap A \in \mathcal{D}$ and $B \in \mathcal{D}'$. Hence $\mathcal{D} = \mathcal{D}'$.

Now consider

$$\mathcal{D}'' = \{B \in \mathcal{D} : B \cap A \in \mathcal{D} \text{ for all } A \in \mathcal{D}\}.$$

Then $\mathcal{A} \subseteq \mathcal{D}''$ because $\mathcal{D} = \mathcal{D}'$. We can check that $\mathcal{D}''$ is a d-system, just as we did for $\mathcal{D}'$. Hence $\mathcal{D}'' = \mathcal{D}$ which shows that $\mathcal{D}$ is a π-system as promised. $\square$