MATHEMATICAL TRIPOS: PART IA **Lent 2017**

PROBABILITY **JRN**

Example Sheet 1 (of 4)

1. Four mice are chosen (without replacement) from a litter, two of which are white. The probability that both white mice are chosen is twice the probability that neither is chosen. How many mice are there in the litter?

2. A table-tennis championship for 2^n players is organized as a knock-out tournament with n rounds, the last round being the final. Two players are chosen at random. Calculate the probability that they meet: (a) in the first round, (b) in the final, (c) in any round.

[*Hint. The same probability space can be used for all three calculations.*]

3. A full deck of 52 cards is divided in half at random. Find an expression for the probability that each half contains the same number of red and black cards. Evaluate this expression as a decimal expansion. Use Stirling's formula to find an approximation for the same probability and evaluate this approximation as a decimal expansion.

4. State what it means for $\mathcal{F}$ to be a σ-algebra and for $\mathbb{P}$ to be a probability measure. Let $(A_n : n \in \mathbb{N})$ be a sequence of events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show, starting from the definitions, the following properties:

(a) $\emptyset$ and $A_1 \cup A_2$ and $\cap_{n=1}^{\infty} A_n$ are events,

(b) $\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(A_1^c) = 1 - \mathbb{P}(A_1)$,

(c) if A_1 and A_2 are disjoint, then $\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2)$,

(d) if $A_1 \subseteq A_2$, then $\mathbb{P}(A_1) \leq \mathbb{P}(A_2)$,

(e) $\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2)$,

(f) if $A_n \subseteq A_{n+1}$ for all n, then $\mathbb{P}(A_n) \to \mathbb{P}(\cup_n A_n)$.

5. (a) Show that, for any three events A, B, C,

$$\mathbb{P}(A^c \cap (B \cup C)) = \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C) - \mathbb{P}(C \cap A) - \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B \cap C).$$

(b) How many of the numbers $1, \ldots, 500$ are not divisible by 7 but are divisible by 3 or 5?

6. Let $(A_n : n \in \mathbb{N})$ be a sequence of events in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Set

$$A = \{\omega \in \Omega : \omega \in A_n \text{ infinitely often}\}, \quad B = \{\omega \in \Omega : \omega \in A_n \text{ for all sufficiently large } n\}.$$

(a) Show that $B = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$.

(b) Show that A is an event and that $\mathbb{P}(A) \leq \sum_{k=n}^{\infty} \mathbb{P}(A_k)$ for all n.

(c) Suppose that the series $\sum_{n=1}^{\infty} \mathbb{P}(A_n)$ converges. Show that $\mathbb{P}(A) = 0$.

7. A committee of size r is chosen at random from a set of n people. Calculate the probability that m given people will all be on the committee (a) directly, (b) using the inclusion-exclusion formula. Deduce that

$$\binom{n-m}{r-m} = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \binom{n-j}{r}.$$

8. Examination candidates are graded into four classes known conventionally as I, II-1, II-2 and III, with probabilities 1/8, 2/8, 3/8 and 2/8 respectively. Candidates who misread the rubric, a common event with probability 2/3, generally do worse, their probabilities being 1/10, 2/10, 4/10 and 3/10. What is the probability:

(a) that a candidate who reads the rubric correctly is placed in the class II-1?

(b) that a candidate who is placed in the class II-1 has read the rubric correctly?

9. Parliament contains a proportion p of Labour members, who are incapable of changing their minds about anything, and a proportion $1-p$ of Conservative members who change their minds completely at random (with probability r) between successive votes on the same issue. A randomly chosen member is noticed to have voted twice in succession in the same way. What is the probability that this member will vote in the same way next time?

10. The Polya urn model for contagion is as follows. We start with an urn which contains one white ball and one black ball. At each second we choose a ball at random from the urn and replace it together with one more ball of the same colour. Calculate the probability that when n balls are in the urn, i of them are white. You might like to carry out a computer simulation. Do you think the proportion of white balls might tend to a limit?

11. Mary tosses two coins and John tosses one coin. What is the probability that Mary gets more heads than John? Answer the same question if Mary tosses three coins and John tosses two. Make a conjecture for the probability when Mary tosses $n+1$ and John tosses n. Can you prove your conjecture?

12. Suppose that n balls are tossed independently and at random into n boxes. What is the probability that exactly one box is empty? Check your answer for $n=2$ and $n=3$ directly.

13. What is the probability that a random non-decreasing function $\{1,\dots,k\} \to \{1,\dots,n\}$ is increasing?

14. Let $(X_n)_{n\geq 0}$ be a simple symmetric random walk on $\mathbb{Z}$, starting from 0.

(a) Show that, for $h = h(n) = 2/\sqrt{n}$, in the limit $n \to \infty$ with n even,

$$\mathbb{P}(X_n = 0) \sim \frac{1}{\sqrt{2\pi}} h.$$

(b) Show further that, for all $x \in \mathbb{R}$,

$$\mathbb{P}(X_n/\sqrt{n} \in [x, x+h)) \sim \frac{1}{\sqrt{2\pi}} h e^{-x^2/2}.$$

[*Hints. For all x and all n, $X_n/\sqrt{n}$ takes exactly one value in $[x, x+h)$. Recall that $(1+1/x)^x \to e$ as $x \to \pm\infty$.*]