9. Full controllability of linear systems

Consider the linear (discrete-time, deterministic controllable dynamical) system:

$$f : \underbrace{\mathbb{R}^d}_{\text{state space}} \times \underbrace{\mathbb{R}^m}_{\text{action space}} \longrightarrow \mathbb{R}^d, \qquad f(x,a) = \underbrace{A}_{d \times d \text{ matrix}} x + \underbrace{B}_{d \times m \text{ matrix}} a.$$

Recall that, given a starting point x_0 and a control $(u_n : n \in \mathbb{Z}^+)$, the controlled sequence $(x_n : n \in \mathbb{Z}^+)$ is defined by

$$x_{n+1} = f(x_n, u_n) = Ax_n + Bu_n, \qquad n \geq 0.$$

We consider whether it is possible to control the linear system $f(x, a) = Ax + Ba$ so as to get from any starting point to any finishing point in n steps. If so, then f is called <u>fully controllable in n steps</u>.

Only the control values $u_0, \dots, u_{n-1}$ are relevant.

The <u>energy</u> of the control is defined to be $\sum_{k=0}^{n-1} |u_k|^2$.

Proposition 9.1

The system $f(x,a) = Ax + Ba$ is fully controllable in n steps if and only if $\operatorname{rank}(M_n) = d$, where

$$M_n = [A^{n-1}B, \dots, AB, B].$$

Moreover, in this case, there is a unique minimal energy control $u = (u_0, \dots, u_{n-1})$ from x_0 to x in n steps, which is given by

$$u_k^T = y^T G_n^{-1} A^{n-k-1} B, \quad k = 0, 1, \dots, n-1,$$

and has energy $y^T G_n^{-1} y$, where

$$y = x - A^n x_0, \quad G_n = M_n M_n^T.$$

Proof By induction on $n \geq 0$, we have

$$x_n = A^n x_0 + A^{n-1} B u_0 + \dots + A B u_{n-2} + B u_{n-1} = A^n x_0 + M_n u$$

where $u = \begin{pmatrix} u_0 \\ \vdots \\ u_{n-1} \end{pmatrix}$. This makes the first assertion clear.

Fix $x_0, x \in \mathbb{R}^d$ and choose u so that $M_n u = y$. By Cauchy-Schwarz

$$y^T G_n^{-1} y = y^T G_n^{-1} M_n u \leq (y^T G_n^{-1} M_n M_n^T G_n^{-1} y)^{\frac{1}{2}} |u|$$

so

$$\sum_{k=0}^{n-1} |u_k|^2 = |u|^2 \geq y^T G_n^{-1} y$$

with equality if and only if $u^T = y^T G_n^{-1} M_n$. $\square$

We now look at a continuous-time analogue.

Thus we look for the first time at a continuous-time controllable dynamical system.

This is specified in general by a function

$$b : S \times A \longrightarrow \mathbb{R}^d$$

where $S \subseteq \mathbb{R}^d$, and for a control $u : \mathbb{R}^+ \longrightarrow A$, the controlled process starting from x_0 is given by

$$\dot{x}_t = b(x_t, u_t).$$

More on the general case later.

For now we consider the case of a linear system

$$b(x, u) = Ax + Bu, \quad x \in \mathbb{R}^d, \quad u \in \mathbb{R}^m.$$

Define

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!}$$

Then

$$(e^{At})^{-1} = e^{-At}, \quad \frac{d}{dt} e^{At} = A e^{At} = e^{At} A.$$

Assume that $u : \mathbb{R}^+ \longrightarrow \mathbb{R}^m$ is piecewise continuous

For a piecewise continuously differentiable function $x : \mathbb{R}^+ \longrightarrow \mathbb{R}^d$, we have

$$\dot{x}_t = Ax_t + Bu_t, \quad t \geq 0,$$

if and only if

$$\frac{d}{dt}\left(e^{-At}x_t\right) = e^{-At}\left(\dot{x}_t - Ax_t\right) = e^{-At}Bu_t, \quad t \geq 0,$$

if and only if

$$x_t = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu_s\,ds, \quad t \geq 0$$

So we see that the plant equation $\dot{x}_t = b(x_t, u_t)$ has an explicit and unique solution.

Define
$$G(t) = \int_0^t e^{As} BB^T (e^{As})^T ds.$$

Lemma 9.2

For all $t > 0$, $G(t)$ is invertible if and only if $\operatorname{rank}(M_d) = d$.

Proof If $\operatorname{rank}(M_d) \le d-1$, there there is a non-zero vector v such that $v^T A^n B = 0$ for all $n = 0, 1, \dots, d-1$, and hence for all n by the Cayley-Hamilton theorem. Then $v^T e^{As} B = 0$ for all $s > 0$ and so $v^T G(t) v = 0$ for all $t > 0$.

On the other hand, if $\operatorname{rank}(M_d) = d$, then, given $v \in \mathbb{R}^d \setminus \{0\}$, $v^T A^n B \neq 0$ for some $n \geq 0$ which we can choose minimal. Then $|v^T e^{As} B| \sim |v^T A^n B| s^n / n!$ as $s \downarrow 0$, so $v^T G(t) v > 0$ for all $t > 0$. $\square$

Say that the linear system $\dot{x}_t = Ax_t + Bu_t$ is <u>fully controllable in time</u> t if

for all $x_0, x \in \mathbb{R}^d$, there is a control $(u_s : 0 \leq s \leq t)$ such that $x_t = x$.

<u>Proposition 9.3</u>

The system $\dot{x}_t = Ax_t + Bu_t$ is fully controllable in time t if and only if $G(t)$ is invertible. In this case, there is a unique minimal energy control from x_0 to x in time t given by $u_s^T = y^T G(t)^{-1} e^{A(t-s)} B$, $0 \leq s \leq t$, with <u>energy</u> $\int_0^t |u_s|^2 ds = y^T G(t)^{-1} y$, where $y = x - e^{At} x_0$.

Proof We have $x_t = x$ if and only if $\int_0^t e^{A(t-s)} B u_s \, ds = y$.
Suppose $G(t)$ is invertible, and set $u_s^{*T} = y^T G(t)^{-1} e^{A(t-s)} B$, $0 \le s \le t$.
Then

$$\int_0^t e^{A(t-s)} B u_s^* \, ds = \int_0^t e^{A(t-s)} B B^T (e^{A(t-s)})^T ds \; G(t)^{-1} y = y,$$

and, by Cauchy-Schwarz, for any control u such that $x_t = x$,

$$y^T G(t)^{-1} y = \int_0^t y^T G(t)^{-1} e^{A(t-s)} B u_s \, ds$$

$$\le \underbrace{\left(\int_0^t y^T G(t)^{-1} e^{A(t-s)} B B^T (e^{A(t-s)})^T G(t)^{-1} y \, ds \right)^{\frac{1}{2}}}_{(y^T G(t)^{-1} y)^{\frac{1}{2}}} \left(\int_0^t |u_s|^2 ds \right)^{\frac{1}{2}}$$

with equality if and only if $u = u^*$.

Suppose on the other hand that $G(t)$ has a null vector $v \neq 0$.

Then

$$0 = v^T G(t) v = \int_0^t v^T e^{As} B B^T (e^{As})^T v \, ds = \int_0^t |v^T e^{As} B|^2 ds$$

so

$$v^T e^{As} B \qquad \text{for all } 0 \leq s \leq t$$

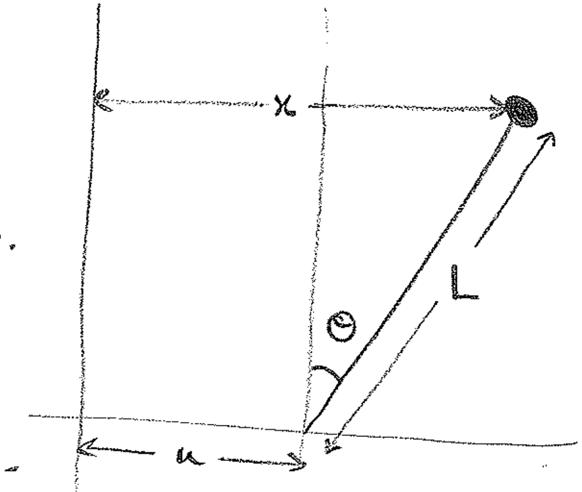
so

$$v^T y = 0 \qquad \text{whenever } y = x - e^{At} x_0,$$

so the system is not fully controllable. $\square$

Since the invertibility of $G(t)$ does not depend on $t > 0$, drop from now on the qualification "in time t" from the notion of full controlability.

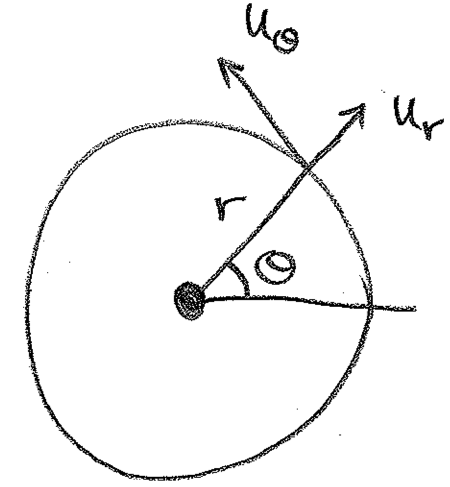
Example – broom balancing



Assume all mass resides in the head.
The only force acting on the broom perpendicular to the stick is gravity.
By Newton's laws

$$g \sin\theta = \ddot{u}\cos\theta + L\ddot{\theta}.$$

Linearize this system near the fixed point $\theta = 0$ and $u = 0$.
Is the linearized system fully controllable?



<u>Example - satellite in a planar orbit</u>

The equations of motion are

$$\ddot{r} = r\dot{\Theta}^2 - \frac{c}{r^2} + u_r, \quad \ddot{\Theta} = -\frac{2\dot{r}\dot{\Theta}}{r} + \frac{u_\Theta}{r}.$$

For each $\rho > 0$, there is a steady-state solution

$$r = \rho, \quad \ddot{\Theta} = \omega = \sqrt{c/\rho^3}.$$

Linearize around this solution.

Show that the linearized system is fully controllable using radial thrust u_r and tangential thrust u_Θ, but not by radial thrust alone.

10. Linear systems with non-negative quadratic costs

Consider the linear system

$$f(x,a) = Ax + Ba, \qquad x \in \mathbb{R}^d, \quad a \in \mathbb{R}^m$$

with quadratic cost function

$$c(x,a) = \begin{pmatrix} x^T & a^T \end{pmatrix} \begin{pmatrix} R & S^T \\ S & Q \end{pmatrix} \begin{pmatrix} x \\ a \end{pmatrix}$$

$$= x^T \underbrace{R}_{d \times d \text{ symmetric}} x + x^T S^T a + a^T \underbrace{S}_{m \times d} x + a^T \underbrace{Q}_{m \times m \text{ symmetric}} a .$$

We assume

$c(x,a) \geq 0$ <u>for all</u> x, a, $\quad Q$ <u>is positive-definite</u>.

Consider the n-horizon problem with non-negative quadratic final cost $V_0(x) = x^T \Pi_0 x$. As usual, set

$$V_n^u(x) = \sum_{k=0}^{n-1} c(x_k, u_k) + V_0(x_n), \qquad V_n(x) = \inf_u V_n^u(x), \qquad n \geq 1,$$

where

$$x_0 = x, \quad x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, \dots, n-1.$$

We saw in §3 that

$$V_{n+1}(x) = \inf_a \{c(x, a) + V_n(Ax + Ba)\}, \qquad n \geq 0.$$

We are going to see now that all these infimal cost functions are quadratic, and that they and the optimal controls are given by an explicit matrix recursion.

Step 1

We compute

$$\frac{\partial}{\partial a} c(x,a) = 2x^T S^T + 2a^T Q$$

So $c(x,\cdot)$ has a unique turning point $a = -Q^{-1}Sx$ which must be its global minimum. Thus

$$\inf_a c(x,a) = c(x, Kx) = x^T(R - S^T Q^{-1} S)x$$

where $K = -Q^{-1}S$.

Step 2

Given a non-negative definite matrix Π, consider

$$\tilde{c}(x,a) = c(x,a) + f(x,a)^T \Pi f(x,a)$$

then

$$\tilde{c}(x,a) = x^T \tilde{R} x + x^T \tilde{S}^T a + a^T \tilde{S} x + a^T \tilde{Q} a$$

where

$$\tilde{R} = R + A^T \Pi A, \quad \tilde{S} = S + B^T \Pi A, \quad \tilde{Q} = Q + B^T \Pi B.$$

Note that $\tilde{c}$ is non-negative and Q is positive definite.

We apply Step 1 to $\tilde{c}$ to see that

$$\inf_a \{ c(x,a) + f(x,a)^T \Pi f(x,a) \} = \tilde{c}(x, K(\Pi)x) = x^T r(\Pi) x,$$

where

$$K(\Pi) = -\tilde{Q}^{-1}\tilde{S} = -(Q + B^T \Pi B)^{-1}(S + B^T \Pi A),$$

$$r(\Pi) = \tilde{R} - \tilde{S}^T \tilde{Q}^{-1} \tilde{S}$$

$$= (R + A^T \Pi A) - (S + B^T \Pi A)^T (Q + B^T \Pi B)^{-1} (S + B^T \Pi A)$$

Proposition 10.1

Define $(\Pi_n)_{n\geq 0}$ by the Riccati recursion $\Pi_{n+1} = r(\Pi_n)$, $n \geq 0$. Then $V_n(x) = x^T \Pi_n x$ and the optimal control for the n-horizon problem, and its controlled sequence, are given by

$$u_k = K(\Pi_{n-k-1}) x_k, \quad x_{k+1} = \Gamma_{n-k-1} x_k, \quad k = 0, 1, \dots, n-1,$$

where $\Gamma_n = A + BK(\Pi_n)$ is the gain matrix.

Proof We have $V_0(x) = x^T \Pi_n x$ and may suppose inductively that $V_n(x) = x^T \Pi_n x$. Then V_{n+1} is given by the optimality equation, and so $V_{n+1}(x) = x^T \Pi_{n+1} x$ by taking $\Pi = \Pi_n$ in Step 2. The induction proceeds.

By Proposition 3.1, the minimizing action in the optimality equation provides an optimal control.

After k steps of the n-horizon problem, we have an $(n-k)$-horizon problem, so, by Step 2 with $\Pi = \Pi_{n-k-1}$,

$$u_k = K(\Pi_{n-k-1}) x_k, \quad x_{k+1} = \Gamma_{n-k-1} x_k. \qquad \square$$

Turn now to the infinite horizon problem. As usual

$$V^u(x) = \sum_{k=0}^{\infty} c(x_k, u_k), \qquad V(x) = \inf_u V^u(x).$$

If f is fully controllable, then there exists a control such that $x_k = 0$ and $u_k = 0$ for all $k \geq d$, so $V(x) < \infty$ for all $x \in \mathbb{R}^d$.

Call a $d \times d$ matrix a <u>stability matrix</u> if $A^n \to 0$ as $n \to \infty$. This holds if and only if all the (complex) eigenvalues of A have modulus less than 1.

Call $f(x,a) = Ax + Ba$ <u>stabilizable</u> if $A + BK$ is a stability matrix for some K.

<u>Example</u> Take $A = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Then $f(x,a) = Ax + Ba$ is not fully controllable.

But $A + (-2 \ \ 0)B = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ is a stability matrix.

The value function of a stabilizable system is finite.

For, if we set $u_n = Kx_n$, then $x_n = \Gamma^n x_0$, where $\Gamma = A + BK$.

So, for all $x \in \mathbb{R}^d$

$$V(x) \leq V^u(x) = x^T \sum_{n=0}^{\infty} (\Gamma^n)^T Q_K \Gamma^n x \leq \frac{|Q_K| |x|^2}{1 - |\Gamma|^2} < \infty$$

where

$$Q_K = \begin{pmatrix} I \\ K \end{pmatrix}^T \begin{pmatrix} R & S^T \\ S & Q \end{pmatrix} \begin{pmatrix} I \\ K \end{pmatrix}.$$

(Here we are writing $|A|$ for the operator norm and are using $|A_1 A_2| \leq |A_1| |A_2|$, $|A^T| = |A|$.)

Proposition 10.2

Assume that f is fully controllable or stabilizable.
Then the infimal cost function is given by $V(x) = x^T \Pi x$, $x \in \mathbb{R}^d$, where Π is the minimal non-negative definite solution to the equilibrium Riccati equation $\Pi = r(\Pi)$, and, for $K = K(\Pi)$, $u(x) = Kx$ defines an optimal control.

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If Q_K is positive-definite, in particular, if c is so, then

- $\Gamma = A + BK$ is a stability matrix,
- Π is the only non-negative definite solution to $\Pi = r(\Pi)$,
- for any non-negative definite matrix Π_0, if we define $\Pi_{n+1} = r(\Pi_n)$ for all $n \geq 0$, then $\Pi_n \to \Pi$ as $n \to \infty$.

Proof. By Proposition 2.1, V satisfies the optimality equation

$$V(x) = \inf_a \{ c(x,a) + V(Ax + Ba) \}, \quad x \in \mathbb{R}^d.$$

Take, for now $\Pi_0 = 0$ and define $\Pi_{n+1} = r(\Pi_n)$, $n \geq 0$.

By Proposition 10.1, with $\Pi_0 = 0$,

$$x^T \Pi_n x = V_n(x) \uparrow V_\infty(x) \leq V(x), \quad x \in \mathbb{R}^d.$$

Since $\mathcal{F}$ is fully controllable or stabilizable, $V_\infty(x) \leq V(x) < \infty$ for all x, so $V_\infty(x) = x^T \Pi x$ for all x, for some Π non-negative definite. Since r is continuous, we can let $n \to \infty$ in $\Pi_{n+1} = r(\Pi_n)$ to see that $\Pi = r(\Pi)$. Then by Step 2 above

$$V_\infty(x) = \min_a \{ c(x,a) + V_\infty(Ax + Ba) \}, \quad x \in \mathbb{R}^d$$

with minimum at $a = u(x) = K(\Pi)x$.

Given $x \in \mathbb{R}^d$, set $x_0 = x$ and $x_{n+1} = u(x_n)$, $n \geq 0$.

We argue as in Proposition 6.1

$$V_\infty(x) = V_n^u(x) + V_\infty(x_n) \geq V_n^u(x) \uparrow V^u(x) \geq V(x).$$

Hence $V(x) = V_\infty(x) = x^T \Pi x$ and u is optimal.

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We have
$$\sum_{n=0}^{\infty} (\Gamma^n)^T Q_K \Gamma^n = \Pi < \infty$$

so, if Q_K is positive-definite, then $\Gamma^n \longrightarrow 0$ as $n \to \infty$.

Consider the n-horizon problem with final cost $x^T \tilde{\Pi}_0 x$.
The associated infimal cost function $\tilde{V}_n(x) = x^T \tilde{\Pi}_n x$, where $\tilde{\Pi}_{n+1} = r(\tilde{\Pi}_n)$.
Then

$$V_n(x) \leq \tilde{V}_n(x) \leq V_n^u(x) + x_n^T \tilde{\Pi}_0 x_n$$

If $r(\tilde{\Pi}_0) = \tilde{\Pi}_0$ then, on letting $n \to \infty$, we obtain $\Pi \leq \tilde{\Pi}_0$,
so Π is the minimal non-negative definite solution.
In the case where Q_K is positive-definite, for general $\tilde{\Pi}_0$,
we have $x_n^T \tilde{\Pi}_0 x_n \to 0$ so

$$x^T \Pi x \leq \lim_{n \to \infty} x^T \tilde{\Pi}_n x \leq x^T \Pi x, \quad x \in \mathbb{R}^d,$$

so $\tilde{\Pi}_n \to \Pi$. In particular Π is the only solution to $\Pi = r(\Pi)$.

10.15 $\square$