14. The LQG model in equilibrium

What happens in the Kalman filter

and in the optimal control problem with quadratic costs

as $n \to \infty$?

Keep adding noise, so $\Sigma_n = \mathrm{var}(\Delta_n) \not\to 0$

But keep taking observations, so may hope Σ_n is bounded.

Consider the LQG system

$$X_{n+1} = AX_n + BU_n + \varepsilon_{n+1}$$
$$Y_{n+1} = CX_n .$$

Assume that the instantaneous costs and the noise are non-degeneration, so $\begin{pmatrix} R & S \\ S^T & Q \end{pmatrix}$, $\begin{pmatrix} N & L^T \\ L & M \end{pmatrix}$ are both positive-definite.

Assume also

stability : $(A+BK)^n \to 0$ as $n \to \infty$ for some K

<u>asymptotic observability</u> : $(A-HC)^n \to 0$ as $n \to \infty$ for some H

Recall (Proposition 9.1 and 10.2)

$$\operatorname{rank}(A^{d-1}B, \dots, AB, B) = d$$

$\Leftrightarrow$ fully controllable $\Rightarrow$ stability

and

$$\operatorname{rank}((A^T)^{d-1}C^T, \dots, A^TC^T, C^T) = d$$

$\Leftrightarrow$ observable.

Hence

observability $\Rightarrow$ asymptotic observability

(Take $B = C$ to see $(A^T - C^TH^T)^n \to 0$ as $n \to \infty$ for some H.)

Theorem 14.1

Under the above assumptions, the equations $\Pi = r(\Pi)$, $\Sigma_1 = \sigma(\Sigma_1)$ both have unique solutions in the set of non-negative definite matrices. Set $H = H(\Sigma_1)$, $K = K(\Pi)$. Define

$$\hat{X}_0 = 0, \quad \hat{X}_{n+1} = A\hat{X}_n + BU_n + H(Y_{n+1} - C\hat{X}_n), \quad U_n = K\hat{X}_n, \quad n \geq 0.$$

Then

$$\mathbb{E}\left(\frac{1}{n} \sum_{k=0}^{n-1} c(X_k, U_k) \right) \longrightarrow \operatorname{trace}(R\Sigma_1) + \operatorname{trace}(\hat{N}\Pi)$$

where $\hat{N} = N + A\Sigma A^T - \Sigma$. Moreover this choice of H, K is optimal.

Sketch of proof

Existence and uniqueness of Π was shown in Proposition 10.2.

Compare the equations

$$\Sigma = g(\Sigma) = (N + A\Sigma A^T) - (L + A\Sigma C^T)(M + C\Sigma C^T)^{-1}(L + A\Sigma C^T)^T$$
$$\Pi = r(\Pi) = (R + A^T\Pi A) - (S + B^T\Pi A)^T(Q + B^T\Pi B)^{-1}(S + B^T\Pi A).$$

Our hypothesis are symmetric under

$$R \to N, \quad S \to L^T, \quad Q \to M, \quad A \to A^T, \quad B \to C^T,$$

So we also get existence and uniqueness of Σ.

We showed in §12 that

$$\mathbb{E}_{(0,\Sigma)} \sum_{k=0}^{n-1} c(X_k, U_k) = n\left\{ \operatorname{trace}(\hat{N}\Pi) + \operatorname{trace}(R\Sigma) \right\}.$$

Left to you:

- as $n \to \infty$, the initial mean and variance become insignificant
- get optimality for H, K from what we know about the n-horizon problem.