Optimization and Control: Examples Sheet 2
LQG models

1. A simple model of the rolling motion of a ship represents it as a damped simple pendulum driven by wave motion. For small roll angles the equation is

\[ \ddot{\theta} + 2\gamma \omega \dot{\theta} + \omega^2 \theta = \omega^2 u, \]

where \( \theta(t) \) is the roll angle and \( u(t) \) is the effective rolling torque from wave motion; \( \omega \) and \( \gamma \) are positive constants.

Show that \( \theta \) and \( \dot{\theta} \) can in principle be moved from any values to any other values in an arbitrary short time by an appropriate control \( u \).

2. Consider a scalar deterministic linear system,

\[ x_{k+1} = Ax_k + Bu_k, \]

with \( n \)-horizon total cost function \( \sum_{k=0}^{n-1} Qu_k^2 + x_n^2 \). By considering a suitable dynamic optimality equation, show that the infimal cost function \( V_n(x) \) has the form \( \Pi_n x^2 \) for some constant \( \Pi_n \), and that the sequence \( \Pi_{n-1} \) obeys a linear recurrence. Hence show that

\[ \Pi_n = \left( \frac{B^2}{Q(A^2 - 1)} + \left( 1 - \frac{B^2}{Q(A^2 - 1)} \right) A^{-2n} \right)^{-1}. \]

Under what conditions does \( \Pi_n \) tend to a limit as \( n \to \infty \)? Investigate the limiting forms of \( \Pi_n \) and of the gain factor \( \Gamma_n \).

3. Successive attempts are made to regulate the speed of a clock, but each deliberate change in setting introduces also a random change whose size tends to increase with the size of the intended change. Explicitly, let \( X_n \) be the error in the speed of the clock after \( n \) corrections. On the basis of the observed value of \( X_n \) one attempts to correct the speed by an amount \( U_n \). The actual error in speed then becomes

\[ X_{n+1} = X_n - U_n + \varepsilon_{n+1} \]

where, conditional on events up to the choice of \( U_n \), the variable \( \varepsilon_{n+1} \) is normally distributed with zero mean and variance \( \alpha U_n^2 \). If, after all attempts at regulation, one leaves the clock with an error \( x \), then there is a cost \( x^2 \).

Suppose exactly \( n \) attempts are to be made to regulate the clock with initial error \( x \). Determine the optimal policy and the minimal expected cost.

4. Consider the scalar-state control problem with plant equation

\[ X_{k+1} = X_k + U_k + \varepsilon_{k+1} \]

and total cost function \( \sum_{k=0}^{n-1} U_k^2 + DX_n^2 \). Here, the initial state \( X_0 = x \) is given, the current state \( X_k \) is observable, the horizon point \( n \) is prescribed, and the disturbances \( \varepsilon_1, \ldots, \varepsilon_n \) are independent with zero mean and common variance \( v \). Find the minimal expected total cost \( i) \) when, for \( k = 0, 1, \ldots, n-1 \), the control \( U_k \) may be chosen to depend on \( X_0, \ldots, X_k \) and \( ii) \) when all controls \( U_k \) must be chosen at the outset. (These are called closed loop and open loop controls respectively.)
5. Consider the real-valued system defined by
\[ X_{n+1} = aX_n + \xi_n U_n \quad (n = 0, 1, \ldots), \]
where \( U_n \) is the control at time \( n \) and \( (\xi_n)_{n \geq 0} \) is a sequence of independent random variables with mean \( b \) and variance \( \sigma^2 \). Suppose that the cost incurred at time \( n \) is \( X_n^2 + U_n^2 \), and that there are no terminal costs. Find the recursions satisfied by the finite-horizon infimal cost functions. Is the optimal control certainty-equivalent?

6. Suppose that a discrete-time system with \( d \)-dimensional state variable \( x \) has a plant equation which is linear in state, \( x_{k+1} = A_k x_k + b(k, u_k) \), an instantaneous cost \( c(k, u_k) \) which is independent of state, and a terminal cost at time \( n \) that is a function of \( d^T x_n \), for a given vector \( d \). Show that the infimal cost function takes the form \( V(k, x) = \phi(k, \xi_k) \), where \( \xi_k = d^T z_k \) is the ‘predicted miss distance’ and \( z_k = A_{n-1} \ldots A_k x_k \) is the the value that \( x_n \) would take starting from \( x_k \) at time \( k \) if we could take \( b = 0 \) from then on. Show that the optimal control at time \( k \) is also a function of \( \xi_k \) and \( k \) alone.

7. Consider the linear system given by \( X_0 = x, V_0 = v \) and, for \( n \geq 0 \),
\[ X_{n+1} = X_n + V_n, \quad V_{n+1} = V_n + U_n + \varepsilon_n. \]
Here \((X_n, V_n)\) are the state variables, representing the position and velocity of a body, \( U_n \) is the control variable, which may be chosen as a function of the history of the system up to time \( n \), and \( (\varepsilon_n)_{n \geq 0} \) is a sequence of independent zero-mean disturbances, with common variance \( \sigma^2 \). The objective is to minimize the expected value of \( \sum_{k=0}^{n-1} U_k^2 + P_0 X_n^2 \). For a given choice of controls, how does \( X_n \) depend on \((x, v)\)? Show that the optimal choice of \( U_0 \) is
\[ U_0 = -(n-1)P_n(x + nv), \]
where
\[ P_n^{-1} = P_0^{-1} + \frac{1}{n}(n-1)(2n-1). \]

8. A one-dimensional model of the problem faced by a juggler trying to balance a light stick with a weight on top is given by the equation
\[ \ddot{x}_1 = \alpha(x_1 - u) \]
where \( x_1 \) is the horizontal displacement of the top of the stick from some fixed point and \( u \) is the horizontal displacement of the bottom. (The stick is assumed to be nearly upright and stationary and \( \alpha > 0 \) is inversely proportional to the length.) Show that the juggler can control \( x_1 \) by manipulating \( u \).

If he tries to balance \( d \) such weighted sticks on top of one another, the equations governing stick \( k \) \((k = 2, \ldots, d)\) are (provided the weights on the sticks get smaller fast enough as \( d \) increases)
\[ \ddot{x}_k = \alpha(x_k - x_{k-1}) \]
Show that the \( d \)-stick system is fully controllable. [You may find it helpful to take the state vector as \((\dot{x}_1, x_1, \dot{x}_2, x_2, \ldots, \dot{x}_d, x_d)^T\).]
9. Consider the following control problem with imperfect state observation

\[ Y_n = CX_n + \eta_n, \quad X_{n+1} = AX_n + BU_n + \varepsilon_{n+1}. \]

Here \( X_0 \sim N(\mu_0, V_0) \) and \( \eta_0, (\varepsilon^1_1, \varepsilon^2_1), \ldots \) are independent non-degenerate zero-mean Gaussians, and \( U_n \) is to be chosen as a function of the observations \( Y_0, \ldots, Y_n \). Compute the conditional distribution of \( X_0 \) given \( Y_0 \). Find a way to transform this problem to the standard LQG model.

10. Consider the controlled system \( X_{n+1} = X_n + U_n + 3\varepsilon_{n+1} \), where \( (\varepsilon_n)_{n \geq 1} \) is a sequence of independent \( N(0, 1) \) variables. The instantaneous cost at time \( n \) is \( X_n^2 + 2U_n^2 \). Assuming that \( X_n \) is observable at time \( n \), calculate the optimal control under steady-state (stationary) conditions and find the expected cost per unit time incurred when this control is used.

Suppose now that one can observe at time \( n \) only \( Y_n = X_{n-1} + 2\eta_n \), for \( n \geq 1 \), where \( (\eta_n)_{n \geq 1} \) is another sequence of independent \( N(0, 1) \) variables, independent of \( (\varepsilon_n)_{n \geq 1} \). Show that, in a steady state, the conditional variance of \( X_n \), given \( (Y_1, \ldots, Y_n) \), is 12.

Determine the optimal control and a recursion for the optimal estimate of state under stationary conditions.