9 Full controllability of linear systems

We begin a detailed study of linear controllable dynamical systems by finding criteria for existence of controls to get from any given state to any other.

Consider the linear controllable dynamical system, with state-space $\mathbb{R}^d$ and action-space $\mathbb{R}^m$, given by

$$f(x,a) = Ax + Ba, \quad x \in \mathbb{R}^d, \quad a \in \mathbb{R}^m.$$

Here A is a $d \times d$ matrix and B is a $d \times m$ matrix. We say that f is *fully controllable in n steps*[22] if, for all $x_0, x \in \mathbb{R}^d$, there is a control $(u_0, \dots, u_{n-1})$ such that $x_n = x$. Here, $(x_0, \dots, x_n)$ is the controlled sequence, given by $x_{k+1} = f(x_k, u_k)$ for $0 \leqslant k \leqslant n-1$. We then seek to minimize the *energy* $\sum_{k=0}^{n-1} |u_k|^2$ over the set of such controls.

Proposition 9.1. *The system f is fully controllable in n steps if and only if* $\operatorname{rank}(M_n) = d$, *where M_n is the $d \times nm$ matrix $[A^{n-1}B, \dots, AB, B]$. Set $y = x - A^n x_0$ and $G_n = M_n M_n^T$. Then the minimal energy from x_0 to x in n steps is $y^T G_n^{-1} y$ and this is achieved uniquely by the control*

$$u_k^T = y^T G_n^{-1} A^{n-k-1} B, \quad 0 \leqslant k \leqslant n-1.$$

Proof. By induction on $n \geqslant 0$ we obtain

$$x_n = A^n x_0 + A^{n-1} B u_0 + \cdots + B u_{n-1} = A^n x_0 + M_n u, \quad u = \begin{pmatrix} u_0 \\ \vdots \\ u_{n-1} \end{pmatrix},$$

from which the first assertion is clear. Fix $x_0, x \in \mathbb{R}^d$ and a control u such that $M_n u = y$. Then, by Cauchy–Schwarz,

$$y^T G_n^{-1} y = y^T G_n^{-1} M_n u \leqslant (y^T G_n^{-1} M_n M_n^T G_n^{-1} y)^{1/2} |u|,$$

so $\sum_{k=0}^{n-1} |u_k|^2 = |u|^2 \geqslant y^T G_n^{-1} y$, with equality if and only if $u^T = y^T G_n^{-1} M_n$. □

Note that $\operatorname{rank}(M_n)$ is non-decreasing in n and, by Cayley–Hamilton[23], is constant for $n \geqslant d$.

Consider now the continuous-time linear controllable dynamical system

$$b(x,u) = Ax + Bu, \quad x \in \mathbb{R}^d, \quad u \in \mathbb{R}^m.$$

Given a starting point x_0, the controlled process for control $(u_t)_{t \geqslant 0}$ is given by the solution of $\dot{x}_t = b(x_t, u_t)$ for $t \geqslant 0$. We say that b is *fully controllable in time t* if, for all $x_0, x \in \mathbb{R}^d$, there exists a control $(u_s)_{0 \leqslant s \leqslant t}$ such that $x_t = x$. We then seek to minimize the *energy* $\int_0^t |u_s|^2 ds$ subject to $x_t = x$. Note that

$$\frac{d}{dt}(e^{-At} x_t) = e^{-At}(\dot{x}_t - Ax_t) = e^{-At} B u_t,$$

[22] This notion is also called *controllability* in accounts where controllable dynamical systems are called something else.

[23] This standard result of linear algebra states that a matrix satisfies its own characteristic equation

so

$$x_t = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu_s ds.$$

Consider for $t \geqslant 0$ the $d \times d$ matrix

$$G(t) = \int_0^t e^{As}BB^T(e^{As})^T ds.$$

Lemma 9.2. *For all $t > 0$, $G(t)$ is invertible if and only if $\mathrm{rank}(M_d) = d$.*

Proof. If $\mathrm{rank}(M_d) \leqslant d-1$, then we can find $v \in \mathbb{R}^d \setminus \{0\}$ such that $v^T A^n B = 0$ for all $n \leqslant d-1$, and hence for all $n \geqslant 0$ by Cayley–Hamilton. Then $v^T e^{As}B = 0$ for all s and so $v^T G(t) v = 0$ for all $t \geqslant 0$. On the other hand, if $\mathrm{rank}(M_d) = d$, then, given $v \in \mathbb{R}^d$, there is a smallest $n \geqslant 0$ such that $v^T A^n B \neq 0$. Then $|v^T e^{As}B| \sim |v^T A^n B| s^n/n!$ as $s \downarrow 0$, so $v^T G(t) v > 0$ for all $t > 0$. □

Proposition 9.3. *The system b is fully controllable in time t if and only if $G(t)$ is invertible. The minimal energy for a control from x_0 to x in time t is $y^T G(t)^{-1} y$, where $y = x - e^{At}x_0$, and is achieved uniquely by the control*

$$u_s^T = y^T G(t)^{-1} e^{A(t-s)} B.$$

The proof is similar to the proof of the discrete-time result and is left as an exercise. As the invertibility of $G(t)$ does not depend on the value of $t > 0$, we speak from now of simply of *full controllability* in the case of continuous time linear systems.

Example (Broom balancing). You attempt to balance a broom upside-down by supporting the tip of the stick in your palm. Is this possible?

We can resolve the dynamics in components to reduce to a one-dimensional problem. Write u for the horizontal distance of the tip from a fixed point of reference, and write θ for angle made by the stick with the vertical. Suppose that all the mass resides in the head of the broom, at a distance L from the tip. Newton's Law gives, for the component perpendicular to the stick of the acceleration of the head

$$g \sin\theta = \ddot{u}\cos\theta + L\ddot{\theta}.$$

We investigate the linearized dynamics near the fixed point $\theta = 0$ and $u = 0$. Replace θ by $\varepsilon\theta$ and u by εu. Then

$$g\varepsilon\theta = \varepsilon\ddot{u} + L\varepsilon\ddot{\theta} + O(\varepsilon^2),$$

so, in terms of $x = u + L\theta$ the linearized system is $\ddot{x} = \alpha(x-u)$, where $\alpha = g/L$, that is,

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = A\begin{pmatrix} x \\ \dot{x} \end{pmatrix} + Bu, \quad A = \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ -\alpha \end{pmatrix}.$$

Then $\mathrm{rank}[AB, B] = 2$, so the linearized system is fully controllable. This provides evidence that, when the broom is close to vertical, we can bring by a suitable choice of control from any initial condition to rest while vertical.

Example (Satellite in a planar orbit). The following equations of motion describe a satellite moving in a planar orbit with radial thrust u_r and tangential thrust u_θ:

$$\ddot{r} = r\dot{\theta}^2 - \frac{c}{r^2} + u_r, \quad \ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} + \frac{u_\theta}{r}.$$

For each $\rho > 0$, there is a solution with $\dot{\theta} = \omega = \sqrt{c/\rho^3}$. We linearize around this solution, setting $r = \rho + \varepsilon x$, $\dot{\theta} = \omega + \varepsilon z$, $u_r \varepsilon u$ and $u_\theta = \varepsilon v$. After some routine calculations, and introducing $y = \dot{x}$, we obtain the linear controllable dynamical system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} + B \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 3\omega^2 & 0 & 2\omega\rho \\ 0 & -2\omega/\rho & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/\rho \end{pmatrix}.$$

It is straightforward to check that $\operatorname{rank}[AB, B] = 3$, so the linear system is fully controllable. On the other hand, if the tangential thrust would fail, so $v = 0$, we would have to replace B by its first column B_1. We have

$$B_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad AB_1 = \begin{pmatrix} 1 \\ 0 \\ -2\omega/\rho \end{pmatrix}, \quad A^2 B_1 = \begin{pmatrix} 0 \\ -\omega^2 \\ 0 \end{pmatrix},$$

so $\operatorname{rank}[A^2B_1, AB_1, B_1] = 2$ and the system is not fully controllable. In fact, it is the angular momentum which cannot be controlled, as

$$\frac{d}{dt}(r^2\dot{\theta}^2) = (2\omega\rho, 0, \rho^2)^T \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}, \quad (2\omega\rho, 0, \rho^2)^T A^n B = 0, \quad n \geqslant 0.$$