18 Existence and uniqueness of solutions for differential equations

The possibility defining a controlled path $(x_t)_{0\leqslant t\leqslant T}$ using a differential equation is assured by the following result, at least in the case where b and u are continuous. The result does not form an examinable part of the course.

Proposition 18.1. *Let $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ be continuous and suppose that, for some $K<\infty$, for all $0\leqslant t\leqslant T$,*

$$|b(t,x)-b(t,y)|\leqslant K|x-y|, \quad x,y\in\mathbb{R}^d. \tag{4}$$

Then, for all $x_0\in\mathbb{R}^d$, there is a unique differentiable function $(x_t)_{0\leqslant t\leqslant T}$ such that

$$\dot{x}_t = b(t,x_t), \quad 0\leqslant t\leqslant T.$$

Proof. Note that, by continuity, $C=\sup_{t\leqslant T}|b(t,x_0)|<\infty$. Set $x_t(0)=x_0$ for all $t\geqslant 0$ and define recursively for $n\geqslant 0$

$$x_t(n+1)=x_0+\int_0^t b(s,x_s(n))ds, \quad 0\leqslant t\leqslant T. \tag{5}$$

Set $f_n(t)=\sup_{s\leqslant t}|x_s(n)-x_s(n-1)|$. Then

$$f_1(t)\leqslant\int_0^t|b(s,x_0)|ds\leqslant Ct, \quad t\leqslant T,$$

and, for $n\geqslant 1$,

$$f_{n+1}(t)=\sup_{s\leqslant t}\left|\int_0^t\{b(s,x_s(n))-b(s,x_s(n-1))\}ds\right|\leqslant K\int_0^t f_n(s)ds.$$

Then, by induction $f_n(t)\leqslant CK^{n-1}t^n/n!$. Hence, $\sum_n f_n(T)<\infty$, so the functions $x(n)$ converge uniformly on $[0,T]$ to a continuous limit x. We can let $n\to\infty$ in (5) to obtain

$$x_t=x_0+\int_0^t b(s,x_s)ds, \quad 0\leqslant t\leqslant T.$$

Since the integrand here is continuous, we deduce that x is differentiable in t and satisfies $\dot{x}_t=b(t,x_t)$ for all t. Finally, if $(y_t)_{t\leqslant T}$ also has this property, we can define $f(t)=\sup_{s\leqslant t}|x_s-y_s|$. Then f is bounded, say by $B<\infty$, so, arguing as above,

$$f(t)\leqslant K\int_0^t f(s)ds\leqslant BK^{n-1}t^n/n!$$

for all n and t. Hence $x_t=y_t$ for all t. □

We remark that the assumption of continuity in t is unnecessarily strong, especially if the differential equation is interpreted in its integrated form. In particular, in several examples we shall wnat to consider the case where b has a discontinuity in t at some time. The (uniform in t) *Lipschitz condition* 4 is implied by a uniform bound on the gradient of b (in x). If it can be seen *a priori* that any solution stays in a given convex subset U of $\mathbb{R}^d$, then it is only necessary to establish such a bound on U.

We can apply the proposition to the control problem provided that we assume b is continuous on $[0,T] \times \mathbb{R}^d \times A$ and satisfies, for some $K < \infty$, for all $0 \leqslant t \leqslant T$ and $a \in A$,

$$|b(t,x,a) - b(t,y,a)| \leqslant K|x-y|, \quad x, y \in \mathbb{R}^d.$$

Then, for any continuous control $u : [0,T] \to A$, the differential equation

$$\dot{x}_t = b^u(t, x_t), \quad 0 \leqslant t \leqslant T,$$

has a unique solution, where $b^u(t,x) = b(t,x,u_t)$.