14 The LQG model in equilibrium

We show that full controllability of the $(A, B, .)$-system, together with observability of the $(A, ., C)$-system, is sufficient for the existence of an equilibrium control in the LQG model. We also discuss the optimal such control.

In Section 9 we showed that the $(A, B, .)$-system is fully controllable if and only if $\text{rank}(M_d) = d$. Also, by Proposition 10.2, this condition implies that the $(A, B, .)$-system is stabilizable, that is, there exists a matrix K such that $|A + BK| < 1$.

In the previous section we saw that the $(A, ., C)$-system is observable if and only if $\text{rank}(N_d) = d$. Now

$$N_d^T = \begin{pmatrix} C^T & C^T A^T & \dots & C^T (A^T)^{d-1} \end{pmatrix},$$

so, by comparing with the form of M_d, we deduce that observability implies the existence of a matrix H such that $|A - HC| = |A^T - C^T H^T| < 1$. We call this last condition *asymptotic observability.*

In the remainder of this section, we consider the LQG model

$$\begin{aligned} X_{n+1} &= AX_n + BU_n + \varepsilon_{n+1}, \\ Y_{n+1} &= CX_n + \eta_{n+1}, & n \geqslant 0, \end{aligned}$$

as in Section 12, and we assume stability and asymptotic observability, that is, the existence of K and H such that $|A + BK| < 1$ and $|A - HC| < 1$. We assume also that both the instantaneous costs and the noise are non-degenerate, that is to say, the matrices

$$\begin{pmatrix} R & S \\ S^T & Q \end{pmatrix}, \quad \begin{pmatrix} N & L^T \\ L & M \end{pmatrix}$$

are both positive-definite.

Theorem 14.1. *Under the above assumptions, the equations $\Pi = r(\Pi)$ and $\Sigma = \sigma(\Sigma)$ both have unique solutions in the set of non-negative-definite matrices. Set $H = H(\Sigma)$ and $K = K(\Pi)$. Define recursively $\hat{X}_0 = 0$ and*

$$U_n = K\hat{X}_n, \quad \hat{X}_{n+1} = A\hat{X}_n + BU_n + H(Y_{n+1} - C\hat{X}_n), \quad n \geqslant 0.$$

Then the long-run average expected cost is given by

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E} \sum_{k=0}^{n-1} c(X_k, U_k) = \text{trace}(R\Sigma) + \text{trace}(\hat{N}\Pi),$$

where $\hat{N} = N + A\Sigma A^T - \Sigma$. Moreover our choice of H and K minimizes the long-run average expected cost.

Outline proof. We showed existence and uniqueness of Π in Proposition 10.2. The existence and uniqueness of Σ can be deduced by comparing the forms of the equations $\Pi = r(\Pi)$ and $\Sigma = \sigma(\Sigma)$. From Section 12, the minimal long-run average expected cost from Δ is $\text{trace}(R\Sigma)$. From Section 11, the minimal long-run average expected cost of the controllable dynamical system for $\hat{X}$ is $\text{trace}(\hat{N}\Pi)$, using control K. □